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# COMPRESSIBLE TURBULENCE

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## ABSTRACT

We present a model to treat fully compressible, nonlocal, time-dependent turbulent convection in the presence of large-scale flows and arbitrary density stratification. The problem is of interest, for example, in stellar pulsation problems, especially since accurate helioseismological data are now available, as well as in accretion disks. Owing to the difficulties in formulating an analytical model, it is not surprising that most of the work has gone into numerical simulations. At present, there are three analytical models: one by the author, which leads to a rather complicated set of equations; one by Yoshizawa; and one by Xiong. The latter two use a Reynolds stress model together with phenomenological relations with adjustable parameters whose determination on the basis of terrestrial flows does not guarantee that they may be extrapolated to astrophysical flows. Moreover, all third-order moments representing nonlocality are taken to be of the down gradient form (which in the case of the planetary boundary layer yields incorrect results). In addition, correlations among pressure, temperature, and velocities are often neglected or treated as in the incompressible case.

To avoid phenomenological relations, we derive the full set of dynamic, time-dependent, nonlocal equations to describe all mean variables, second- and third-order moments. Closures are carried out at the fourth order following standard procedures in turbulence modeling. The equations are collected in an Appendix.

Some of the novelties of the treatment are (1) new flux conservation law that includes the large-scale flow, (2) increase of the rate of dissipation of turbulent kinetic energy owing to compressibility and thus (3) a smaller overshooting, and (4) a new source of mean temperature due to compressibility; moreover, contrary to some phenomenological suggestions, the adiabatic temperature gradient depends only on the thermal pressure, while in the equation for the large-scale flow, the physical pressure is the sum of thermal plus turbulent pressure.

*Subject headings:* hydrodynamics — turbulence

## 1. INTRODUCTION

In this paper, we deal with a fully compressible flow described by velocity and temperature fields $u_i$ and $T$, which we take as

$$u_i = \tilde{u}_i + u_i'' , \qquad T = \tilde{T} + T'' . \tag{1a}$$

The density $\rho$ and pressure $p$, related by a perfect gas equation of state

$$p = R\rho T , \tag{1b}$$

are split as

$$p = \bar{p} + p', \qquad \rho = \bar{\rho} + \rho' . \tag{1c}$$

Here, an overbar and/or angle brackets denote ensemble average. We include body forces (e.g., gravity) and a radiative field whose form will be left unspecified. As explained in § 2, we use the mass average process to treat the fields $\boldsymbol{u}$ and $T$. We derive the following results:

First-order moments :

1. Dynamical equation for the mean density $\bar{\rho}$ in terms of the large-scale velocity field $\tilde{\boldsymbol{u}}$.
2. Dynamical equation for the large-scale flow $\tilde{\boldsymbol{u}}$. This entails a second-order moment representing the flux of the turbulent velocity field, the Reynolds stresses $R_{ij}$.
3. Dynamical equation for the mean temperature field, $\tilde{T}$. This entails a second-order moment representing the flux of the fluctuating temperature field, the enthalpy, or convective flux, $H_i$.

Second-order moments:

1. Dynamical equation for the Reynolds stresses, $R_{ij} = \bar{\rho}^{-1}\overline{\rho u_i'' u_j''}$.
2. Dynamical equation for the enthalpy/convective flux, $H_i = \bar{\rho}^{-1}c_p \overline{\rho u_i'' T''}$.
3. Dynamical equation for the temperature variance, $\Psi = \frac{1}{2}\bar{\rho}^{-1}\overline{\rho T''^2}$.

Third-order moments:

The equations for $R_{ij}$, $H_i$, and $\Psi$ entail third-order moments of the type

$$R_{ijk} = \bar{\rho}^{-1}\overline{\rho u_i'' u_j'' u_k''} , \qquad H_{ij} \equiv \bar{\rho}^{-1}\overline{u_i'' u_j'' T''} , \qquad \Psi_i \equiv \bar{\rho}^{-1}\overline{\rho u_i'' T''^2} , \tag{1d}$$

which are known as diffusion terms since they appear as divergences, and

$$\Pi_{ij} = \overline{u_i'' \frac{\partial p'}{\partial x_j}} + \overline{u_j'' \frac{\partial p'}{\partial x_i}} , \qquad \Pi_i^\theta \equiv \overline{T'' \frac{\partial p'}{\partial x_i}} , \tag{1e}$$

which represent pressure correlations. Rather than using phenomenological expressions or assuming their incompressible counterparts, we derive the dynamic, time-dependent equations for equations (1d) and (1e). The latter, in turn, entail higher order moments

$$\Pi_{ijk} = \overline{u_i'' u_j'' \frac{\partial p'}{\partial x_k}} , \qquad \Pi_{ij}^\theta = \overline{u_i'' T'' \frac{\partial p'}{\partial x_j}} , \qquad \Pi_i^{\theta\theta} = \overline{\rho'^2 \frac{\partial p'}{\partial x_i}} , \tag{1f}$$

for which we also derive the corresponding dynamic, time-dependent equations.

In addition, compressibility acts as a source of dissipation $\epsilon$, which is now the sum of a solenoidal (incompressible) and a dillation (compressible) part, $\epsilon_s$ and $\epsilon_d$, which must be modeled. In addition, we work out the expression for the new variables

$$\bar{u}_i'' , \overline{T''} , \overline{p' u_i''} , \overline{p' d} , \overline{u_i'' \frac{Dp}{Dt}} , \overline{T'' \frac{Dp}{Dt}} , \tag{1g}$$

which enter in several of the dynamic equations. The complete set of equations contains only one assumption, that $\rho'$, $p'$, and $\rho T''$ satisfy a polytropic relation.

In the Appendix, we summarize all the relevant equations.

## 2. REYNOLDS AND MASS AVERAGES

As in previous work (Lele 1994; Favre 1969; Canuto 1992, 1993, 1994, 1996; Rubesin 1989, 1990; Sarkar & Balakrishnan 1990; Taulbee & VanOsdol 1991; Speziale & Sarkar 1991; Sarkar 1992; Sarkar et al. 1989, 1993), pressure and density are written as

$$p = \bar{p} + p' , \qquad \rho = \bar{\rho} + \rho' , \tag{2a}$$

with the general rule that a stochastic variable $\xi$

$$\xi = \bar{\xi} + \xi' \tag{2b}$$

satisfies the following relations

$$\bar{\xi} \equiv \langle \xi \rangle , \qquad \langle \xi' \rangle \equiv \bar{\xi'} = 0 . \tag{2c}$$

In the case of compressible turbulence, it is more appropriate to treat the other variables via a "mass average" process, whereby

$$\xi = \tilde{\xi} + \xi'' , \tag{3a}$$

where

$$\tilde{\xi} \equiv \{\xi\} = \frac{\langle \rho\xi \rangle}{\langle \rho \rangle} , \qquad \langle \rho\xi'' \rangle = 0 . \tag{3b}$$

Thus, an overbar and/or angle brackets represent a Reynolds average, while a tilde and/or curly brackets represent a mass average. We stress that in either case we are dealing with statistical averages (time averages if one adopts the ergodic hypothesis), in contrast to the volume averages that are used in numerical approaches such as LES, large eddy simulation. As we have said, the velocity and temperature fields are written as

$$u_i = \tilde{u}_i + u_i'' , \qquad T = \tilde{T} + T'' . \tag{3c}$$

With the above definitions, we derive the following relations:

$$\langle \xi'' \rangle = -\frac{\langle \rho' \xi'' \rangle}{\langle \rho \rangle} , \tag{3d}$$

$$\langle \rho' \xi'' \rangle = \langle \rho' \xi' \rangle , \tag{3e}$$

$$\langle \xi'' \rangle = \langle \xi \rangle - \frac{\langle \rho\xi \rangle}{\langle \rho \rangle} \equiv \langle \xi \rangle - \{\xi\} \equiv \langle \xi \rangle - \tilde{\xi} \tag{3f}$$

It is important to stress that

$$\langle \xi'' \rangle \neq 0 , \qquad \langle \xi' \rangle = 0 . \tag{3g}$$

Using an equation of state of the form ($R = 1$)

$$p = \rho T , \tag{4a}$$

we derive

$$\bar{p} = \overline{\rho T} = \bar{\rho}\,\frac{\overline{\rho T}}{\bar{\rho}} = \bar{\rho}\tilde{T}\,, \tag{4b}$$

$$p' = \rho'\tilde{T} + \rho T''\,. \tag{4c}$$

With the Reynolds average and $p = \bar{p} + p'$, $T = \bar{T} + T'$, we have

$$\bar{p} = \bar{\rho}\bar{T} + \overline{\rho' T'}\,, \tag{4d}$$

$$p' = \rho'\bar{T} + \rho T' - \overline{\rho' T'}\,, \tag{4e}$$

which are considerably less intuitive than equations (4b) and (4c). Applying the above rules, we derive the following relations:

$$\langle \rho u_i'' \rangle \equiv \overline{\rho u_i''} = 0\,, \qquad \tilde{u}_i = \frac{\langle \rho u_j \rangle}{\langle \rho \rangle}\,, \tag{5a}$$

$$\langle u_i'' \rangle = -\frac{\langle \rho' u_i'' \rangle}{\langle \rho \rangle} = -\frac{\langle \rho' u_i' \rangle}{\langle \rho \rangle}\,, \tag{5b}$$

$$\langle \rho T'' \rangle \equiv \overline{\rho T''} = 0\,, \qquad \tilde{T} = \frac{\langle \rho T \rangle}{\langle \rho \rangle}\,, \tag{5c}$$

$$\langle T'' \rangle = -\frac{\langle \rho' T'' \rangle}{\langle \rho \rangle} = -\frac{\langle \rho' T' \rangle}{\langle \rho \rangle}\,. \tag{5d}$$

For example, from equations (3f) with $\xi \equiv T$ and (3d), (5d), or from (4b) and (4d), it follows that

$$\tilde{T} = \bar{T} + \frac{\overline{\rho' T'}}{\bar{\rho}} = \bar{T} - \langle T'' \rangle\,, \tag{5e}$$

and thus one may expect that the two types of averages coincide in the incompressible case. Equation (37b) for $\overline{T''}$ confirms that expectation. With these premises, we shall consider several basic equations.

## 3. CONTINUITY EQUATION

Given the general equation governing the density $\rho$

$$\frac{d\rho}{dt} + \rho\,\frac{\partial}{\partial x_i}\,u_i = 0\,, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\,(\rho u_i) = 0\,, \qquad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + u_i\,\frac{\partial}{\partial x_i}\,, \tag{6a}$$

we obtain, upon averaging,

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j}\,(\bar{\rho}\tilde{u}_j) = 0\,, \tag{6b}$$

or alternatively

$$\frac{D}{Dt}\,\bar{\rho} + \bar{\rho}\,\frac{\partial}{\partial x_j}\,\tilde{u}_j = 0\,, \qquad \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \tilde{u}_j\,\frac{\partial}{\partial x_j}\,. \tag{6c}$$

In the stationary case, equation (6b) expresses conservation of the mass flux $\bar{\rho}\tilde{u}_j$. Subtracting equation (6c) from equation (6a), one derives the equation for $\rho'$:

$$\frac{D}{Dt}\,\rho' + \rho'\,\frac{\partial}{\partial x_j}\,\tilde{u}_j + \frac{\partial}{\partial x_j}\,(\rho u_j'') = 0\,, \tag{6d}$$

which can be further transformed into a equation for $\rho'/\bar{\rho}$:

$$\bar{\rho}\,\frac{D}{Dt}\,\frac{\rho'}{\bar{\rho}} + \frac{\partial}{\partial x_i}\,(\rho u_i'') = 0\,. \tag{6e}$$

As expected, taking averages of equation (6e) and making use of equation (2c) and the first of equation (5a), we obtain an identity.

## 4. MOMENTUM EQUATIONS

Consider the Navier-Stokes equations

$$\frac{\partial}{\partial t}\,\rho u_i + \frac{\partial}{\partial x_j}\,\rho u_i u_j = F_i\,, \tag{7a}$$

$$F_i \equiv -\frac{\partial p}{\partial x_i} - \rho g_i + F_i^{\text{vis}} , \qquad F_i^{\text{vis}} \equiv \frac{\partial}{\partial x_j} \sigma_{ij} , \tag{7b, c}$$

where $\sigma_{ij}$ is the viscous stress tensor defined by

$$\sigma_{ij} = \mu\left(\frac{\partial}{\partial x_j} u_i + \frac{\partial}{\partial x_i} u_j\right) - \frac{2}{3} \mu \delta_{ij} \frac{\partial}{\partial x_k} u_k . \tag{7d}$$

Here, $\mu = \nu\rho$ is the dynamic viscosity. Substituting equation (3c) for $u_i$ and equation (2a) for $\rho$, averaging and using the above relations, we derive the following results.

## 5. LARGE-SCALE VELOCITY FIELD $\tilde{\boldsymbol{u}}$: DYNAMIC EQUATIONS

Averaging equation (7a), we obtain the dynamic equation for the large-scale flow $\tilde{\boldsymbol{u}}$,

$$\frac{\partial}{\partial t} (\bar{\rho}\tilde{u}_i) + \frac{\partial}{\partial x_j} (\bar{\rho}\tilde{u}_i \tilde{u}_j + \tau_{ij}) = \bar{F}_i \tag{8a}$$

or

$$\bar{\rho} \frac{D}{Dt} \tilde{u}_i = \bar{F}_i - \frac{\partial}{\partial x_j} \tau_{ij} . \tag{8b}$$

In equations (8a) and (8b) $\tau_{ij}$ are the turbulent Reynolds stresses

$$\tau_{ij} \equiv \overline{\rho u_i'' u_j''} = \bar{\rho}\{u_i'' u_j''\} . \tag{8c}$$

Since the field $\tilde{\boldsymbol{u}}$ represents the large-scale flow, it is justified to assume that it is not affected by viscosity, and so

$$\bar{F}_i = -\frac{\partial}{\partial x_i} \bar{p} - \bar{\rho} g_i . \tag{8d}$$

Thus, finally,

$$\bar{\rho} \frac{D}{Dt} \tilde{u}_i = -\frac{\partial}{\partial x_j} (\bar{p}\, \delta_{ij} + \tau_{ij}) - \bar{\rho} g_i \tag{8e}$$

## 6. LARGE-SCALE VELOCITY FIELD $\tilde{\boldsymbol{u}}$: STRESSES

For the large-scale velocity field $\tilde{\boldsymbol{u}}$, the Reynolds stress tensor can be defined as

$$t_{ij} \equiv \tilde{u}_i \tilde{u}_j . \tag{9a}$$

Multiplying equation (8b) for $\tilde{u}_i$ by $\tilde{u}_j$ and repeating the operation with $i$ and $j$ interchanged, we obtain, upon summing the two equations,

$$\bar{\rho} \frac{D}{Dt} \tilde{u}_i \tilde{u}_j = -(\tilde{u}_i \tau_{jk,k} + \tilde{u}_j \tau_{ik,k}) + \bar{F}_i \tilde{u}_j + \bar{F}_j \tilde{u}_i . \tag{9b}$$

As expected, the space derivatives of the turbulent stresses $\tau_{ij}$ act as a source of $t_{ij}$ much as the space derivatives of the large-scale flow (shear) act as a source of $\tau_{ij}$. Using equation (8d), we further have

$$\bar{F}_i \tilde{u}_j + \bar{F}_j \tilde{u}_i = -\left(\frac{\partial \bar{p}}{\partial x_k} + \bar{\rho} g_k\right)(\tilde{u}_i \delta_{jk} + \tilde{u}_j \delta_{ik}) . \tag{9c}$$

To solve equations (8a) and/or (9b), one needs to know the turbulent Reynolds stress $\tau_{ij}$. Equation (9b) will be used in § 13 to derive the generalized Bernoulli's equation. However, we must note that the field $\tilde{\boldsymbol{u}}$ is obtained through the solution of equation (8e).

## 7. TURBULENT VELOCITY FIELD

Consider equation (7a) written in the equivalent form,

$$\frac{\partial}{\partial t} u_i + u_j \frac{\partial}{\partial x_j} u_i = \rho^{-1} F_i . \tag{10a}$$

Substitute $u_i$ from equation (3c) and subtract equation (8b). The result is the equation for $u_i''$:

$$\frac{D}{Dt} u_i'' + u_j'' \tilde{u}_{i,j} + u_j'' u_{i,j}'' = \rho^{-1} F_i - (\bar{\rho})^{-1} \bar{F}_i + (\bar{\rho})^{-1} \tau_{ij,j} , \tag{10b}$$

which can be compared with the equation for the incompressible case, equation (31) of Canuto (1992). In writing equation (10b), we have employed the notation

$$a_{,j} \equiv \partial a/\partial x_j\,; \qquad a_{i,j} \equiv \partial a_i/\partial x_j\,; \qquad a_{ij,k} \equiv \partial a_{ij}/\partial x_k\,. \tag{10c}$$

As a consistency check, one can work out the equation for the vector $q_i \equiv \overline{\rho' u_i''}$. Multiply equation (10b) by $\rho'$ and equation (6d) by $u_i''$. Adding the results and averaging, one obtains

$$\frac{D}{Dt}\,q_i + q_i\tilde{u}_{j,j} + q_j\tilde{u}_{i,j} + \tau_{ij,j} - \bar{\rho}\overline{u_j'' u_{i,j}''} = \overline{\frac{\rho'}{\rho}\,F_i}\,. \tag{10d}$$

The fifth term can be evaluated by averaging equation (10b). Substituting into equation (10d), one obtains

$$\frac{D}{Dt}\,q_i + q_i\tilde{u}_{j,j} + q_j\tilde{u}_{i,j} + \bar{\rho}\,\frac{D}{Dt}\,\overline{u_i''} + \bar{\rho}\overline{u_j''}\,\tilde{u}_{i,j} = 0\,, \tag{10e}$$

which can be rewritten as

$$\frac{D}{Dt}\,(q_i + \bar{\rho}\overline{u_i''}) + (q_j + \bar{\rho}\overline{u_j''})\tilde{u}_{i,j} + (q_i + \bar{\rho}\overline{u_i''})\tilde{u}_{j,j} = 0\,, \tag{10f}$$

where we have eliminated $D\bar{\rho}/Dt$ via the first of equation (6c). Since

$$q_i + \bar{\rho}\overline{u_i''} = \overline{\rho u_i''} \tag{10g}$$

is zero in the mass average process, equation (5a), equation (10f) is identically satisfied.

## 8. TURBULENT FIELD: REYNOLDS STRESSES

To derive the dynamic equation for the Reynolds stresses (eq. [8c]), we multiply equation (7a) for $u_i$ by $u_j$ and multiply equation (7a) for $u_j$ by $u_i$. Adding the two equations, we obtain

$$\frac{\partial}{\partial t}\,(\rho u_i u_j) + \frac{\partial}{\partial x_k}\,(\rho u_i u_j u_k) = F_i u_j + F_j u_i\,. \tag{11a}$$

Averaging and making use of the relations

$$\overline{\rho u_i u_j} = \bar{\rho}\tilde{u}_i\tilde{u}_j + \tau_{ij}\,, \tag{11b}$$

$$\overline{\rho u_i u_j u_k} = \bar{\rho}\tilde{u}_i\tilde{u}_j\tilde{u}_k + \tau_{ij}\tilde{u}_k + \tau_{jk}\tilde{u}_i + \tau_{ik}\tilde{u}_j + \tau_{ijk}\,. \tag{11c}$$

as well as of equation (9b), we derive the desired dynamic equation for $\tau_{ij}$, namely,

$$\frac{D}{Dt}\,\tau_{ij} + D_f = S_{ij} + \overline{F_i u_j} - \bar{F}_i\tilde{u}_j + \overline{F_j u_i} - \bar{F}_j\tilde{u}_i\,. \tag{11d}$$

Here, $D_f$ denotes the *diffusion* of $\tau_{ij}$:

$$D_f \equiv \frac{\partial}{\partial x_k}\,\tau_{ijk}\,, \qquad \tau_{ijk} \equiv \overline{\rho u_i'' u_j'' u_k''} \equiv \bar{\rho}R_{ijk}\,. \tag{11e}$$

The tensor $S_{ij}$ denotes the *source* of $\tau_{ij}$ due to the field $\tilde{\boldsymbol{u}}$, that is,

$$-S_{ij} \equiv \tau_{ik}\tilde{u}_{j,k} + \tau_{jk}\tilde{u}_{i,k} + \tau_{ij}\tilde{u}_{k,k}\,, \tag{11f}$$

while the last four terms in equation (11d) comprise pressure distribution, density-velocity correlations, and dissipation terms, which we consider next.

## 9. PRESSURE AND GRAVITY FORCES

Using equations (7b) and (7c), we derive

$$\overline{F_i u_j} + \overline{F_j u_i} = -(\bar{p}_{,k} + \bar{\rho}g_k)(\tilde{u}_i\delta_{jk} + \tilde{u}_j\delta_{ik}) - (\delta_{ik}\overline{u_j''} + \delta_{jk}\overline{u_i''})\bar{p}_{,k} - \left(\overline{u_i''\frac{\partial p'}{\partial x_j}} + \overline{u_i''\frac{\partial p'}{\partial x_j}}\right) + \overline{F_i^{\rm vis}u_j} + \overline{F_j^{\rm vis}u_i}\,. \tag{12a}$$

Since the first term is just equation (9c), we finally have

$$\overline{F_i u_j} + \overline{F_j u_i} - (\bar{F}_i\tilde{u}_j + \bar{F}_j\tilde{u}_i) = B_{ij} - \Pi_{ij} + X_{ij}\,, \tag{12b}$$

where we have defined the three tensors

$$B_{ij} \equiv \frac{1}{\bar{\rho}}\,(\overline{\rho' u_j''}\,\delta_{ik} + \overline{\rho' u_i''}\,\delta_{jk})\bar{p}_{,k}\,, \tag{12c}$$

$$\Pi_{ij} \equiv \overline{u_i'' p'_{,j}} + \overline{u_j'' p'_{,i}} \equiv \Lambda_{ij} + \Lambda_{ji} , \tag{12d}$$

$$X_{ij} \equiv \overline{F_i^{\text{vis}} u_j} + \overline{F_j^{\text{vis}} u_i} . \tag{12e}$$

In writing $B_{ij}$, we have used equation (5b).

## 10. VISCOUS TERMS

First, we compute the tensor $X_{ij}$ in equation (12e). Using equation (7c), we first write

$$X_{ij} = \frac{\partial}{\partial x_k} (\overline{\sigma_{ik} u_j} + \overline{\sigma_{jk} u_i}) - \epsilon_{ij} , \tag{13a}$$

where the "dissipation tensor" $\epsilon_{ij}$ is defined by

$$\epsilon_{ij} = \overline{\sigma_{ik} u_{j,k}} + \overline{\sigma_{jk} u_{i,k}} . \tag{13b}$$

The "diffusive" component of $X_{ij}$ will be included in the diffusion term $D_{ij}$ (see below), while the "dissipative" component $\epsilon_{ij}$ will be discussed in § 14.

## 11. REYNOLDS STRESSES

Putting together equations (12b), (13a), and (13b), equation (11d) becomes

$$\bar{\rho}\left[\frac{D}{Dt} R_{ij} + D_{ij}\right] = \Sigma_{ij} + B_{ij} - \pi_{ij} + \delta_{ij} PD - \epsilon_{ij} . \tag{14a}$$

We have redefined the Reynolds stresses, diffusion tensor, source term, and pressure-velocity correlation as follows:

$$R_{ij} \equiv \bar{\rho}^{-1} \tau_{ij} = \bar{\rho}^{-1} \overline{\rho u_i'' u_j''} , \tag{14b}$$

$$D_{ij} = \bar{\rho}^{-1} \frac{\partial}{\partial x_k} \left[ \bar{\rho} R_{ijk} + \frac{2}{3} \delta_{ij} \overline{p' u_k''} - \overline{\sigma_{ik} u_j} - \overline{\sigma_{jk} u_i} \right] , \tag{14c}$$

$$R_{ijk} \equiv \bar{\rho}^{-1} \tau_{ijk} = \bar{\rho}^{-1} \overline{\rho u_i'' u_j'' u_k''} , \tag{14d}$$

$$-\Sigma_{ij} \equiv \bar{\rho}[R_{ik} \tilde{u}_{j,k} + R_{jk} \tilde{u}_{i,k}] , \tag{14e}$$

$$\pi_{ij} \equiv \Pi_{ij} - \tfrac{1}{3} \delta_{ij} \Pi_{kk} . \tag{14f}$$

The pressure-dilatation term $PD$ is defined as

$$PD = \tfrac{2}{3} \overline{p' u_{i,i}''} \equiv \tfrac{2}{3} \overline{p' d} , \tag{14g}$$

where $d$ is the "dilatation" defined as

$$d = \frac{\partial}{\partial x_i} u_i'' \equiv u_{i,i}'' , \tag{14h}$$

while $B_{ij}$ and $\Pi_{ij}$ are still given by equations (12c) and (12d). The physical interpretation of equation (14a) is as follows: $\Sigma_{ij}$ represents a source term due to the shear of the large-scale flow $\tilde{\boldsymbol{u}}$, $B_{ij}$ represents a source due to mass fluctuations (see the interpretation as a buoyancy term in the next section), $\Pi_{ij}$ represents the contribution of pressure gradients, while the last two terms represent dilation effects and viscous dissipation. The expressions for $B_{ij}$, $PD$, and $\epsilon_{ij}$ will be given in § 14, $\Pi_{ij}$ will be discussed in § 15, while the nonlocal, third-order diffusion term will be discussed in § 16.

## 12. TURBULENT KINETIC ENERGY AND TURBULENT PRESSURE

To help understand the physical content of equation (14a), consider the equation for the turbulent kinetic energy $K$:

$$K \equiv \tfrac{1}{2} \bar{\rho}^{-1} \overline{\rho u_i'' u_i''} \tag{15a}$$

Taking the trace of equation (14a), we obtain

$$\bar{\rho}\left[\frac{D}{Dt} K + D(K)\right] = \frac{1}{2} \Sigma_{ii} + \frac{1}{2} B_{ii} + \overline{p' d} - \bar{\rho}\epsilon , \tag{15b}$$

where $D(K)$ is the diffusion of kinetic energy

$$D(K) \equiv \bar{\rho}^{-1} \frac{\partial}{\partial x_i} \left[ \frac{1}{2} \bar{\rho} R_{kki} + \overline{p' u_i''} - \overline{\sigma_{ij} u_j} \right] \tag{15c}$$

and

$$\frac{1}{2} B_{ii} \equiv (\bar{\rho})^{-1} \overline{\rho' u_i''} \frac{\partial \bar{p}}{\partial x_i} . \tag{15d}$$

Here, $\epsilon$ is the rate of dissipation of kinetic energy (see eq. [27a]),

$$\epsilon_{ij} = \tfrac{2}{3}\bar{\rho}\epsilon\,\delta_{ij} . \tag{15e}$$

From equation (8c) with $i = 3$, $u_3'' \equiv w$, we define a turbulent pressure

$$p_t \equiv \overline{\rho w^2} , \qquad P_t = \bar{\rho}^{-1} p_t . \tag{16a}$$

From equation (14a) we have

$$\bar{\rho}\left[\frac{D}{Dt} P_t + D(P_t)\right] = \Sigma_{33} + B_{33} - \pi_{33} + \frac{2}{3}(\overline{p'd} - \bar{\rho}\epsilon) \tag{16b}$$

with

$$\bar{\rho}D(P_t) = \frac{\partial}{\partial x_i}\left[\bar{\rho}R_{33i} + \frac{2}{3}\overline{p'u_i''} - 2\,\overline{\sigma_{3i}u_3}\right] . \tag{16c}$$

We note, as a matter of illustration, that if we adopt the Boussinesq approximation (Canuto 1992)

$$\rho' \sim -\alpha\bar{\rho}T'' , \qquad \alpha \sim \tilde{T}^{-1} , \qquad \overline{\rho' u_i''} \approx -\alpha\bar{\rho}\overline{u_i'' T''} , \tag{16d}$$

and the hydrostatic equilibrium equation

$$\frac{\partial \bar{p}}{\partial x_i} = -g_i \bar{\rho} , \tag{16e}$$

equation (15d) becomes

$$\frac{1}{2} B_{ii} = (\bar{\rho})^{-1}\overline{\rho' u_i''}\frac{\partial \bar{p}}{\partial x_i} \approx \alpha\bar{\rho} g_i \overline{u_i'' T''} \approx \alpha c_p^{-1} g_i F_i^c , \tag{16f}$$

where $F_i^c$ is the convective flux; see equation (17b). Apart from notational difference, this is the source of turbulent kinetic energy, equation (60) of Canuto (1992).

Finally, we note that two new dissipation terms appear in equation (14a) that are absent in the incompressible case, that is,

$$\overline{p'd} , \qquad \bar{\rho}\epsilon_d , \tag{16g}$$

since, as we shall discuss in § 14,

$$\epsilon = \epsilon_S(\text{incompressible}) + \epsilon_d(\text{compressible}) . \tag{16h}$$

It has been estimated from DNS data that the *additional sources of dissipation* (eq. [16g]) *due entirely to compressibility* can modify the kinetic energy budget by as much as 25%.

## 13. TEMPERATURE FIELD

Next, we consider the temperature field

$$T = \tilde{T} + T'' , \qquad \overline{\rho T} = \bar{\rho}\tilde{T} , \qquad \overline{\rho T''} = 0 , \tag{17a}$$

and derive the dynamic equations for the mean temperature $\tilde{T}$, the temperature flux

$$F_i^c = c_p \overline{\rho T'' u_i''} , \qquad H_i \equiv \bar{\rho}^{-1} F_i^c , \tag{17b}$$

and the temperature variance (which is related to the potential energy)

$$\psi = \tfrac{1}{2}\overline{\rho T''^2} , \qquad \Psi \equiv \bar{\rho}^{-1}\psi . \tag{17c}$$

### 13.1. *Mean Temperature*

We begin with the dynamic equation for total energy

$$h + \tfrac{1}{2}u^2 , \tag{18a}$$

where $h = c_v T + p/\rho$ is the enthalpy and $\frac{1}{2}u^2 \equiv \frac{1}{2}u_i u_i$ is the kinetic energy (per unit mass) where $\boldsymbol{u}$ is the velocity field. We have

$$\rho\frac{d}{dt}\left(h + \frac{1}{2}u^2\right) = \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_j}(\sigma_{ij}u_i) - \rho g_i u_i - \frac{\partial}{\partial x_i}F_i^r , \tag{18b}$$

where $F_i^r$ represents a radiative flux of arbitrary form. Since from Navier-Stokes equations (7a), we have

$$\rho\frac{d}{dt}\frac{1}{2}u^2 = -u_i\frac{\partial p}{\partial x_i} + u_i\frac{\partial}{\partial x_j}\sigma_{ij} - \rho g_i u_i , \tag{18c}$$

we subtract equation (18c) from equation (18b) to obtain the equation for the enthalpy

$$\frac{\partial}{\partial t}\rho h + \frac{\partial}{\partial x_j}(\rho h u_j) = \frac{dp}{dt} + \sigma_{ij}\frac{\partial}{\partial x_j}u_i - \frac{\partial}{\partial x_i}F_i^r , \tag{18d}$$

where we have used the fact that

$$\rho\frac{dh}{dt} \equiv \rho\left(\frac{\partial h}{\partial t} + u_j\frac{\partial h}{\partial x_j}\right) = \frac{\partial}{\partial t}\rho h + \frac{\partial}{\partial x_j}(\rho h u_j) . \tag{18e}$$

Next, we average equation (18d). For a perfect gas, $h = c_v T + p\rho^{-1} = c_p T$, we obtain

$$\overline{\rho h} = c_p \bar{\rho}\tilde{T} , \tag{19a}$$

$$\overline{\rho h u_j} = c_p(\bar{\rho}\tilde{T}\tilde{u}_j + \overline{\rho h'' u_j''}) = c_p\bar{\rho}\tilde{T}\tilde{u}_j + F_j^c , \tag{19b}$$

$$\overline{\frac{dp}{dt}} = \frac{D\bar{p}}{Dt} + \overline{u_i''}\frac{\partial\bar{p}}{\partial x_i} + \overline{u_i''\frac{\partial p'}{\partial x_i}} , \tag{19c}$$

and from equations (13b) and (15e),

$$\overline{\sigma_{ij}u_{j,i}} = \bar{\rho}\epsilon . \tag{19d}$$

The dynamic equation for the mean temperature $\tilde{T}$ is then

$$\bar{\rho}c_p\frac{D\tilde{T}}{Dt} = -\frac{\partial}{\partial x_i}(F_i^c + \overline{F_i^r} - \overline{p'u_i''}) + \frac{D\bar{p}}{Dt} + \overline{u_i''}\,\bar{p}_{,i} - \overline{p'd} + \bar{\rho}\epsilon . \tag{20a}$$

Because of equation (5b), the third term in equation (20a) can be written in terms of the tensor $B_{ij}$ of equation (12c) as

$$\overline{u_i''}\frac{\partial\bar{p}}{\partial x_i} = -\frac{1}{\bar{\rho}}\overline{\rho' u_i''}\frac{\partial\bar{p}}{\partial x_i} = -\frac{1}{2}B_{ii} . \tag{20b}$$

It is important to stress that in equation (20a), $\epsilon$ acts as a source of temperature while it acts as a sink for the turbulent kinetic energy (eq. [15b]). Because of equation (16h), *compressibility acts as a source of mean temperature.* Also, the pressure-dilation effects act in opposite ways in the kinetic energy equation (15b) and in the enthalpy equation (20a). The expressions for the terms $\overline{u_i''}$, $\epsilon$, and $\overline{p'd}$ will be given in §§ 14 and 15.

### 13.2. *Generalized Bernoulli Equation*

In this section, we limit our considerations to an inviscid fluid. Adding equation (20a) to (15b) yields

$$\frac{D\bar{p}}{Dt} = \bar{\rho}R_{ij}\tilde{u}_{i,j} + \bar{\rho}\frac{D}{Dt}[c_p\tilde{T} + K] + \frac{\partial}{\partial x_i}(F_i^c + \bar{F}_i^r + F_i^{\mathrm{ke}}) , \tag{20c}$$

where $F_i^{\mathrm{ke}}$ is the flux of turbulent kinetic energy

$$F_i^{\mathrm{ke}} = \tfrac{1}{2}\overline{\rho u_k'' u_k'' u_i''} . \tag{20d}$$

Next, we add to both sides of equation (20c) the term

$$\bar{\rho}\frac{D}{Dt}\frac{1}{2}\tilde{u}_i\tilde{u}_i \equiv \bar{\rho}\frac{D\tilde{K}}{Dt} , \tag{20e}$$

where $\tilde{K}$ is the kinetic energy of the large-scale flow $\tilde{\boldsymbol{u}}$. Using the trace of equation (9b), equation (20c) becomes

$$\bar{\rho}\frac{D}{Dt}[c_p\tilde{T} + K + \tilde{K}] + \frac{\partial}{\partial x_i}(F_i^c + \bar{F}_i^r + F_i^{\mathrm{ke}} + \bar{\rho}R_{ij}\tilde{u}_j) = \frac{D\bar{p}}{Dt} + \bar{F}_i\tilde{u}_i . \tag{20f}$$

Next, we use the form of $\bar{F}_i$, equation (8d), and recall that

$$g_i\bar{\rho}\tilde{u}_i = \bar{\rho}\tilde{u}_i\frac{\partial G}{\partial x_i} = \bar{\rho}\frac{DG}{Dt} , \tag{20g}$$

since as a rule the gravitational field $G$ does not depend on time. Thus, equation (20f) becomes

$$\frac{\partial\bar{p}}{\partial t} = \bar{\rho}\frac{D}{Dt}(c_p\tilde{T} + K + \tilde{K} + G) + \frac{\partial}{\partial x_i}(F_i^c + \bar{F}_i^r + F_i^{\mathrm{ke}} + \bar{\rho}R_{ij}\tilde{u}_j) . \tag{20h}$$

Equation (20h) is the generalized Bernoulli's equation to include turbulence and radiation.

### 13.3. *New Flux Conservation Law*

In the stationary case, equation (20h) yields the conservation law

$$F_i^c + \bar{F}_i^r + F_i^{\mathrm{ke}} + \bar{\rho}\tilde{u}_j[(c_p\tilde{T} + K + \tilde{K} + G)\delta_{ij} + R_{ij}] = \text{constant} . \tag{20i}$$

What is conserved is the sum of the convective, radiative, turbulent kinetic energy fluxes plus the flux by the large-scale flow of enthalpy, kinetic energy of both turbulence and large-scale flow, gravitational energy, and Reynolds stresses. In stellar structure studies, equation (20i) is usually written as

$$F_i^c + \bar{F}_i^r = \text{constant} , \tag{20j}$$

or, at most, with the inclusion of $F^{ke}$. It is, however, clear that in pulsation problems, the last terms proportional to $\tilde{\boldsymbol{u}}$ cannot be neglected; we recall that $\tilde{\boldsymbol{u}}$ contains not only a time-independent part but also large-scale oscillations, sound waves.

### 13.4. *Convective (Enthalpy) Flux*, $F_i^c = c_p \overline{\rho u_i'' T''} = \bar{\rho} H_i$

To construct the dynamic equation governing the convective flux (eq. [17b]), we begin with equation (18d) written as

$$c_p\left[\frac{\partial}{\partial t}\rho T + \frac{\partial}{\partial x_j}(\rho u_j T)\right] = \frac{dp}{dt} + X \tag{21a}$$

with

$$X \equiv \sigma_{ij} u_{j,i} - F_{i,i}^r . \tag{21b}$$

Use of the continuity equation (6a) and $p = R\rho T$, as well as equation (21a) to eliminate the time dependence of $T$, gives

$$\frac{dp}{dt} = \gamma\left(X - p\frac{\partial u_i}{\partial x_i}\right) - X , \tag{21c}$$

which can be considered the dynamic equation for the pressure $p$. The temperature equation (21a) can thus be written as

$$\frac{\partial}{\partial t}\rho T + \frac{\partial}{\partial x_j}(\rho u_j T) = (A, B) , \tag{21d}$$

where $(A, B)$ means that one can use either $A$ or $B$, where

$$A \equiv c_p^{-1}\left[\frac{dp}{dt} + X\right], \qquad B \equiv c_v^{-1}\left[-p\frac{\partial u_i}{\partial x_i} + X\right] . \tag{21e}$$

Multiplying equation (21d) by $u_i$ and equation (7a) by $T$ and summing, we obtain

$$\frac{\partial}{\partial t}(\rho T u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j T) = F_i T + u_i(A, B) . \tag{22a}$$

Next, we average equation (22a) and employ the results:

$$\overline{\rho T u_i} = \bar{\rho}\tilde{u}_i \tilde{T} + \overline{\rho u_i'' T''} = \bar{\rho}\tilde{u}_i\tilde{T} + c_p^{-1}F_i^c , \tag{22b}$$

$$\overline{\rho u_i u_j T} = \bar{\rho}\tilde{u}_i\tilde{u}_j\tilde{T} + \tilde{T}\tau_{ij} + c_p^{-1}\tilde{u}_k(\delta_{ik}F_j^c + \delta_{kj}F_i^c) + \overline{\rho u_i'' u_j'' T''} , \tag{22c}$$

where $\tau_{ij}$ is the Reynolds stress, equation (8c). Substitute equations (22b) and (22c) into the averaged form of equation (22a) and make use of equation (8a). The result is

$$\frac{D}{Dt}F_i^c + \bar{\rho}D_i = -c_p\tau_{ij}\frac{\partial\tilde{T}}{\partial x_j} + C_i(\tilde{\boldsymbol{u}}) + c_p[\overline{F_i T} - \tilde{T}\bar{F}_i + \overline{u_i(A, B)}] . \tag{23a}$$

Here, $C_i(\tilde{\boldsymbol{u}})$ the contribution of the large-scale field $\tilde{\boldsymbol{u}}$:

$$-C_i(\tilde{\boldsymbol{u}}) \equiv F_i^c\tilde{u}_{j,j} + F_j^c\tilde{u}_{i,j} + \tilde{u}_i\left(c_p\bar{\rho}\frac{D\tilde{T}}{Dt} + \frac{\partial}{\partial x_j}F_j^c\right) , \tag{23b}$$

and $D_i$ represents the diffusion of the convective flux,

$$\bar{\rho}D_i \equiv c_p\frac{\partial}{\partial x_j}h_{ij} , \qquad h_{ij} \equiv \overline{\rho u_i'' u_j'' T''} \equiv \bar{\rho}H_{ij} , \tag{23c}$$

a third-order moment whose form will be given in § 16. Next, we work out the last three terms in equation (23a). Recalling that $\overline{(p')}_{,i} = 0$, we obtain

$$\overline{F_i T} - \tilde{T}\bar{F}_i = -\overline{T''}\frac{\partial\bar{p}}{\partial x_i} - \overline{T''\frac{\partial p'}{\partial x_i}} = -g\bar{\rho}\Lambda_i\overline{T''} - \Pi_i^\theta + \overline{T'' F_i^{vis}} , \tag{23d}$$

where

$$\Lambda_i \equiv H_p\frac{1}{\bar{p}}\frac{\partial\bar{p}}{\partial x_i} , \qquad H_p = \frac{\bar{p}}{g\bar{\rho}} , \qquad \Pi_i^\theta = \overline{T''\frac{\partial p'}{\partial x_i}} . \tag{23e}$$

$H_p$ is the pressure scale height, and $\Pi_i^\theta$ is the pressure-temperature correlation. The expression for $\langle T'' \rangle$ is given below, equation (37b). Using the definition of $A$, equation (21e), we derive

$$c_p \overline{u_i A} = (\tilde{u}_i + \overline{u_i''})\bar{p}_{,t} + \overline{u_i'' \frac{\partial p'}{\partial t}} + [\tilde{u}_i(\tilde{u}_j + \overline{u_j''}) + \tilde{u}_j \overline{u_i''} + r_{ij}]\bar{p}_{,j} + \tilde{u}_i \overline{u_j'' \frac{\partial p'}{\partial x_j}} + \tilde{u}_j \overline{u_i'' \frac{\partial p'}{\partial x_j}} - \tilde{u}_i \overline{F^r_{j,j}} - \overline{u_i'' F^r_{j,j}} + \overline{u_i'' u_j'' \frac{\partial p'}{\partial x_j}} + \overline{u_i \sigma_{jk} u_{k,j}} , \tag{23f}$$

where

$$r_{ij} \equiv \overline{u_i'' u_j''} = R_{ij} - m\tilde{T}^{-1} H_{ij} . \tag{23g}$$

Using the definition of $C_i(\tilde{\boldsymbol{u}})$ and equation (20a), equation (23a) then becomes

$$\bar{\rho}\left[\frac{D}{Dt} H_i + c_p \frac{\partial}{\partial x_j} H_{ij}\right] = c_p \bar{\rho} R_{ij} \beta_j - \bar{\rho} H_j \tilde{u}_{i,j} - c_p H_{ij}\left[\bar{\rho}_{,j} + \left(\frac{m}{c_p \tilde{T}}\right)\bar{p}_{,j}\right] - c_p[g\bar{\rho}\Lambda_i \overline{T''} + \Pi_i^\theta]$$
$$+ \pi_{ijk}\delta_{jk} + \overline{u_i''}\frac{D\bar{p}}{Dt} + \overline{u_i'' \frac{Dp'}{Dt}} - \overline{u_i'' F^r_{j,j}} , \tag{24a}$$

where the higher order tensor

$$\pi_{ijk} \equiv \overline{u_i'' u_j'' \frac{\partial p'}{\partial x_k}} \tag{24b}$$

will be given below, § 16, equations (51g) and (51h). The superadiabatic temperature gradient is defined as

$$\beta_i \equiv -\frac{\partial \tilde{T}}{\partial x_i} + (\bar{\rho} c_p)^{-1} \bar{p}_{,i} = -\frac{\partial \tilde{T}}{\partial x_i} + \frac{g}{c_p}\Lambda_i . \tag{24c}$$

As one can see, the pressure that enters the definition of $\beta$ is not the total pressure, $p$(thermal) + $p$(tur), as has often been suggested on phenomenological grounds. Such renormalization occurs only in equation (8e), which, in the absence of a large-scale flow, becomes the hydrostatic equilibrium equation.

As for the viscous terms, the quantity $\bar{\rho}\epsilon$ brought about through equation (20a) cancels exactly, leaving the terms $\overline{u_i'' \sigma_{jk} u_{k,j}}$ and $\overline{T'' F_i^{\rm vis}}$, which we consider smaller than the last term in equation (24a) which represents the rate of dissipation of $H_i$. Both the gradient of the mean temperature and the gradient of the large-scale flow (shear) act as sources of $H_i$. The expression for $\overline{T''}$, $\overline{T'' p'_{,i}}$, $\overline{u_i''}$, $\overline{u_i'' Dp'/Dt}$ will be given in §§ 14 and 15. The radiation term must be treated in accordance with the chosen model for $F_i^r$. It is, however, important to stress the role of compressibility. Regardless of the model chosen, we can write

$$F_i^r = \tilde{F}_i^r + F_i^{r''} \tag{24d}$$

and thus

$$\overline{u_i'' F^r_{j,j}} = \overline{u_i''}\,\tilde{F}^r_{j,j} + \overline{u_i'' F^{r''}_{j,j}} . \tag{24e}$$

The first term is zero in an incompressible treatment, while the second term is nonzero in both treatments since physically it represents the damping of the convective flux owing to radiative processes. If one employs a representation of the type ($\chi \equiv c_p \rho K_r$, where $K_r$ is the radiative conductivity)

$$F_i^r = -\chi \frac{\partial T}{\partial x_i} , \tag{24f}$$

one can write

$$-\overline{u_i'' F^{r''}_{j,j}} = \frac{1}{2}\chi \frac{\partial^2}{\partial x_j^2} F_i^c , \tag{24g}$$

which becomes important when the radiative timescale

$$\tau_\chi \sim l^2 \chi^{-1} \tag{24h}$$

becomes of the same order as or shorter than the buoyancy timescale. The term (24g) becomes important for small Peclet numbers, that is, when convection is inefficient.

### 13.5. *Temperature Variance* $\psi \equiv \frac{1}{2}\overline{\rho T''^2} = \bar{\rho}\Psi$

Using equations (21d) and (7a), the equation for $\rho T^2$ is

$$\frac{\partial}{\partial t}\rho T^2 + \frac{\partial}{\partial x_j}(\rho T^2 u_j) = 2T(A, B) . \tag{25a}$$

Averaging and using the fact that

$$\overline{\rho T^2} = \bar{\rho}\tilde{T}^2 + \overline{\rho T''^2} , \tag{25b}$$

$$\overline{\rho T^2 u_j} = \bar{\rho}\tilde{u}_j \tilde{T}^2 + \tilde{u}_j \overline{\rho T''^2} + \overline{\rho u''_j T''^2} + 2\tilde{T} c_p^{-1} F_j^c , \tag{25c}$$

we obtain the equation for $\psi$

$$\frac{D}{Dt}\psi + D_f = -\tilde{u}_{i,i}\psi - c_p^{-1} F_i^c \frac{\partial \tilde{T}}{\partial x_i} - c_p^{-1}\tilde{T}\left(c_p \bar{\rho}\frac{D\tilde{T}}{Dt} + \frac{\partial}{\partial x_i} F_i^c\right) + \overline{T(A, B)} , \tag{26a}$$

where the diffusion $D_f$ is defined as

$$D_f \equiv \frac{1}{2}\frac{\partial}{\partial x_i}\psi_i , \qquad \psi_i \equiv \overline{\rho u''_i T''^2} = \bar{\rho}\Psi_i \tag{26b}$$

Using the first of equations (21e) and (20a), equation (26a) becomes

$$c_p\left(\frac{D}{Dt}\psi + D_f + \tilde{u}_{i,i}\psi\right) = -\bar{\rho} H_i \tilde{T}_{,i} + \bar{T}''\frac{D\bar{p}}{Dt} + \overline{T''\frac{Dp'}{Dt}} + \overline{u''_i T'' p_{,i}} - \overline{T'' F^r_{i,i}} , \tag{26c}$$

since

$$\overline{u''_i T'' p_{,i}} = \overline{u'' T''_i}\,\bar{p}_{,i} + \overline{u''_i T'' p'_{,i}} = \left(\frac{g}{c_p}\right)\bar{\rho}\Lambda_i H_i + \frac{1}{2}\Pi^\theta_{ii} , \tag{26d}$$

where

$$\Pi^\theta_{ij} \equiv \left\langle T'' u''_i \frac{\partial p'}{\partial x_j}\right\rangle + \left\langle T'' u''_j \frac{\partial p'}{\partial x_i}\right\rangle \tag{26e}$$

is a pressure-velocity-temperature correlation that will be studied in § 16. Using the second of equation (26b), we finally obtain

$$c_p\bar{\rho}\left(\frac{D\Psi}{Dt} + \bar{\rho}^{-1} D_f\right) = \bar{\rho} H_i \beta_i + \frac{1}{2}\Pi^\theta_{ii} + \bar{T}''\frac{D\bar{p}}{Dt} + \overline{T''\frac{Dp'}{Dt}} - \overline{T'' F^r_{i,i}} . \tag{26f}$$

As expected on physical grounds, the large-scale flow $\tilde{\boldsymbol{u}}$ does not act directly as source of temperature variance while the temperature gradient does.

## 14. CLOSURES

### 14.1. *Compressible Dissipation: Two Models*

Since dissipation by molecular forces occurs at the smallest scales that are nearly isotropic, it has been a standard approximation to assume that $\epsilon_{ij}$ is of the form,

$$\epsilon_{ij} = \tfrac{2}{3}\bar{\rho}\epsilon\,\delta_{ij} . \tag{27a}$$

Using equation (13b), we obtain

$$\bar{\rho}\epsilon = \overline{\sigma_{ij} u_{i,j}} . \tag{27b}$$

Using the definition of $\sigma_{ij}$, equation (7d), one obtains after several steps, the following exact result:

$$\bar{\rho}\epsilon = \mu\left[\overline{\omega''_i \omega''_i} + \frac{4}{3}\overline{d^2}\right] + 2\mu\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_i}\overline{u''_i u''_j} - 2\overline{du''_j}\right] , \tag{27c}$$

where the vorticity $\omega''_i = \epsilon_{ijk}\omega''_{jk}$, $2\omega''_{ij} \equiv u''_{i,j} - u''_{j,i}$, $\epsilon_{ijk}$ being the Levi-Civita tensor. At this point, it is usually assumed that owing to the homogeneity of the small scales, the last two terms can be neglected in comparison with the first two terms. However, this is not a required approximation, since one can think of including the last two terms in the diffusion term. The important fact is that, contrary to the incompressible case, the dissipation $\epsilon$ is now contributed by two terms, a solenoidal (incompressible) and a dilation (compressible) component (Sarkar et al. 1989; Zeman 1990, 1991):

$$\epsilon = \epsilon_s + \epsilon_d , \qquad \epsilon_s = \nu\overline{\omega''_i \omega''_i} , \qquad \epsilon_d = \tfrac{4}{3}\nu\overline{d^2} . \tag{27d}$$

We present two models:

1. In this model, $\epsilon$ is treated as a single variable satisfying a dynamic equation that is an extension of the incompressible case (Canuto 1992). The suggested equation is

$$\frac{D\epsilon}{Dt} + D_f(\epsilon) = -C_{\epsilon 1}\epsilon K^{-1} R_{ij}\tilde{u}_{i,j} - C_{\epsilon 2}\epsilon^2 K^{-1} + C_{\epsilon 4} m(1-\gamma^{-1}) H_p^{-1}\epsilon K^{-1} H_i \Lambda_i + C_{\epsilon 3}\bar{\rho}^{-1}\epsilon K^{-1}\overline{p'd} - \epsilon\tilde{u}_{i,i} , \tag{28a}$$

where $K$ is the turbulent kinetic energy, equation (15a), and

$$-\bar{\rho} D_f(\epsilon) = \frac{\partial}{\partial x_j}\left[C_\epsilon \bar{\rho} K\epsilon^{-1} R_{ij}\frac{\partial \epsilon}{\partial x_i}\right] . \tag{28b}$$

We also recall the identity

$$\bar{\rho}\,\frac{DA}{Dt}=\frac{\partial}{\partial t}\,(\bar{\rho}A)+\frac{\partial}{\partial x_j}\,(\bar{\rho}A\tilde{u}_j)\,. \tag{28c}$$

If we compare equation (28a) with equation (46) of Canuto (1992), we see that the first three terms were already present in the incompressible case, the $C_{\epsilon 4}$ term being the source of dissipation due to buoyancy. Because of $m<0$ and $\Lambda_i<0$, the term is positive in the region of unstable stratification. The last two terms in equation (28a) are due to compressibility.

2. In this model $\epsilon_s$ and $\epsilon_d$ are modeled independently. First, $\epsilon_s$ is taken to satisfy a differential equation similar to the one in the incompressible case, to wit,

$$\frac{D\epsilon_s}{Dt}+D_f(\epsilon_s)=-\,C_{\epsilon 1}\,\epsilon_s K^{-1}R_{ij}\left[\tilde{u}_{i,j}-\frac{1}{3}\,\delta_{ij}\tilde{u}_{k,k}\right]-C_{\epsilon 2}\,\epsilon_s^2\,K^{-1}+C_{\epsilon 4}\,m(1-\gamma^{-1})H_p^{-1}\epsilon_s K^{-1}H_i\Lambda_i-\frac{4}{3}\,\epsilon_s\,\tilde{u}_{i,i}\,. \tag{28d}$$

The form of the diffusion $D_f(\epsilon_s)$ is the same as equation (28b) with $\epsilon\to\epsilon_s$. For $\epsilon_d$, we write

$$\epsilon_d=\epsilon_s\,F(M)\,,\qquad M^2=\frac{2K}{\gamma RT}=2Kc_s^{-2}\,, \tag{28e}$$

where $M$ is the Mach number and $c_s$ is the speed of sound $(=\gamma\bar{p}/\bar{\rho})^{1/2}$. Sarkar et al. (1989) have suggested $F(M)=\alpha_1\,M^2$, with $\alpha_1$ of order unity, while Zeman (1990) has suggested a slightly more complex expression:

$$F(M)=1-\exp\,\{-\,[(M-0.1)/0.6]^2\}\,,\quad M>0.1\quad F=0\,,\quad M<0.1\,. \tag{28f}$$

The constants are $C_{\epsilon 4}=1.44$, $C_{\epsilon 2}=1.83$, $C_{\epsilon 3}=0.15$, and $C_{\epsilon 4}=0.1$ (Sarkar et al. 1989).

### 14.2. *Pressure-Dilatation $\overline{p'd}$, and Dilatational-Dissipation $\epsilon_d$: Old and New Models*

1. Sarkar et al. (1993) have suggested the closure

$$\overline{p'd}=\tfrac{1}{5}\bar{\rho}M^2[\alpha_3\,\epsilon_s+\alpha_2\,R_{ij}\,\tilde{u}_{i,j}] \tag{29a}$$

with $\alpha_{2,3}$ of order unity. Thus, using equation (28e),

$$\overline{p'd}-\bar{\rho}\epsilon_d=-\,[\tfrac{1}{5}(5\alpha_1-\alpha_3)\epsilon_s-\tfrac{1}{5}\alpha_2\,R_{ij}\,\tilde{u}_{i,j}]\bar{\rho}M^2\,. \tag{29b}$$

2. Zeman (1991) has suggested a model valid for $M\le 1$, whereby

$$\overline{p'd}=-\,\frac{1}{2}(\bar{\rho}c_s^2)^{-1}\,\frac{D}{Dt}\,\overline{p'^2}\,, \tag{29c}$$

$$\frac{D}{Dt}\,\overline{p'^2}=-\,\tau_a^{-1}(\overline{p'^2}-p_e^2)\,, \tag{29d}$$

where the acoustic timescale $\tau_a$ and the equilibrium pressure $p_e$ are given by

$$\tau_a=\tau M[54(1+\tfrac{1}{3}M^2)]^{-1/2}\,,\qquad \tau\equiv 2K\epsilon_s^{-1}\,, \tag{29e}$$

$$p_e=2\rho K\phi^{1/2}(M)\,, \tag{29f}$$

$$\phi(M)\equiv\frac{1+2M^2}{1+M^2+2M^4}\,. \tag{29g}$$

The acoustic timescale $\tau_a$ is defined as $\tau_a=L/c$, where $c$ is the propagation of speed of density or pressure fluctuations given by (Chandrasekhar 1951)

$$c=2^{1/2}c_s(1+\tfrac{1}{3}M^2)^{1/2}\,. \tag{30a}$$

Since the turbulent timescale $\tau$ is defined as $\tau=2K\epsilon_s^{-1}$ and $\epsilon_s\sim K^{3/2}L^{-1}$, the first of equation (29e) follows. The function $\phi$ was constructed using two ingredients: (1) the ratio of compressible to solenoidal kinetic energies $K(c)/K(s)$ can be written as the sum of two known behaviors (Sarkar et al. 1989, 1993; Zeman 1990, 1991): $\sim M$ for $M<1$ and $\sim M^4$ for $M\approx O(1)$, so that

$$\frac{K(c)}{K(s)}=M^2(1+2M^2)\,; \tag{30b}$$

and (2) the ratio between compressible potential and kinetic energies is in equilibrium

$$\left(\frac{p_e}{\rho c_s}\right)^2\approx 2K(c)\,. \tag{30c}$$

With $K=K(c)+K(s)$, equation (29f) follows. We may also note that equation (29f) implies that

$$\frac{p_e^2}{\bar{p}^2}=\gamma^2M^4\phi(M)\,. \tag{30d}$$

The novel physical feature of the model is that in order to solve equation (29d) one needs to know the initial pressure $p_0$. This is distinct from the incompressible case in which the time development of the turbulence is governed only by the initial values of the energy and dissipation rates. Direct numerical simulation, DNS (Lele 1994; Lee, Lele, & Moin 1992; Passot & Pouquet 1987; Huang, Coleman, & Bradshaw 1995; Muthsam et al. 1995; Cambon, Coleman, & Mansour 1992), has shown that in the case of compressible turbulence, this is not the case and that for an initial Mach number, the evolution of a turbulent state depends on the level of initial pressure (density) fluctuations as well as on the ratio of compressible to total kinetic energy.

3. The third model relies on the fact that using Poisson equation one can show that $\overline{p'd}$ and $\epsilon_d$ are related by (Taulbee & VanOsdol 1991)

$$\overline{p'd} - \bar{\rho}\epsilon_d = -\bar{\rho}\epsilon_s M^{-2}[C_0 M(\tau\tilde{u}_{k,k})^2 - C_1(\tau\tilde{u}_{k,k}) + C_2]\Gamma , \tag{31a}$$

where

$$\Gamma = \bar{\rho}^{-2}\overline{\rho'^2} \tag{31b}$$

and $C_0 = 1$, $C_1 = \frac{1}{2}$, and $C_2 = \frac{2}{3}$. To complete the model, one needs an equation for $\Gamma$. Previous authors (Rubesin 1989, 1990) have suggested the closure $T'' = Au_i''\beta_i$, where $\beta_i$ is defined in equation (24c). Using a polytrope, equation (35a), we have

$$\Gamma = m^2 A^2 \tilde{T}^{-2} R_{ij}\beta_i\beta_j . \tag{32}$$

From dimensional considerations, $A$ must be of the form $A = c_\epsilon \tau_*$, where $\tau_*$ is a timescale. We suggest two choices, $\tau_* = \tau$ and $\tau_* = \tau_a$; see equation (29e). A value $c_\epsilon = \frac{1}{3}$ is also suggested.

### 14.3. *New Model for* $\overline{p'^2}$ *and* $\epsilon_d$

We begin by deriving a dynamic equation for $\overline{p'^2}$. To that end, we use equation (6d) written for $p'$ via the second of equation (35b), multiply the result by $p'$, and average. We obtain

$$\frac{1}{2}\frac{D\overline{p'^2}}{Dt} = \overline{p'^2}\Phi - \frac{n}{\gamma}c_s^2\left\langle p'\frac{\partial}{\partial x_i}\rho u_i''\right\rangle , \tag{33a}$$

where

$$\Phi \equiv c_s^{-2}\frac{D}{Dt}c_s^2 - \tilde{u}_{k,k} . \tag{33b}$$

Next, we have

$$\overline{p'\frac{\partial}{\partial x_i}\rho u_i''} = \frac{\partial}{\partial x_i}\overline{p'\rho u_i''} - \overline{\rho u_i''\frac{\partial p'}{\partial x_i}} \tag{33c}$$

and

$$\overline{p'\rho u_i''} = \bar{\rho}\overline{p'u_i''} + \overline{\rho' p' u_i''} = \bar{\rho}\overline{p'u_i''} + nm^2\bar{p}(\bar{\rho}\tilde{T}^2)^{-1}\overline{\rho^2 u_i'' T''^2} . \tag{33d}$$

We approximate the last term as follows:

$$\bar{\rho}^{-1}\overline{\rho^2 u_i'' T''^2} = \overline{\rho u_i'' T''^2} = \bar{\rho}\Psi_i , \tag{33e}$$

where $\Psi_i$ was defined in equation (26b), and its dynamical equation is given by equations (62a) and (62b). Thus,

$$\overline{p'\rho u_i''} = \bar{\rho}(\overline{p'u_i''} + nm^2\bar{p}\tilde{T}^{-2}\Psi_i) . \tag{33f}$$

The second term in equation (33c) is given by

$$\overline{\rho u_i''\frac{\partial p'}{\partial x_i}} = \bar{\rho}\overline{u_i''\frac{\partial p'}{\partial x_i}} + \overline{\rho' u_i''\frac{\partial p'}{\partial x_i}} = \frac{1}{2}\bar{\rho}\Pi_{ii} + m\tilde{T}^{-1}\overline{\rho T'' u_i''\frac{\partial p'}{\partial x_i}} , \tag{33g}$$

where $\Pi_{ij}$ is defined in equation (12d) and it will be given in § 15. Finally, if we take $\rho \to \bar{\rho}$ in the last term, we obtain

$$\overline{\rho u_i''\frac{\partial p'}{\partial x_i}} = \frac{1}{2}\bar{\rho}(\Pi_{ii} + m\tilde{T}^{-1}\Pi_{ii}^\theta) , \tag{33h}$$

where $\Pi_{ij}^\theta$ was defined in equation (26e). Finally, the equation for $\langle p'^2\rangle$ is given by

$$\frac{1}{2}\frac{D\overline{p'^2}}{Dt} = \overline{p'^2}\Phi - \frac{n}{\gamma}c_s^2\left[\frac{\partial}{\partial x_i}\bar{\rho}(\overline{p'u_i''} + nm^2\bar{p}\tilde{T}^{-2}\Psi_i) - \frac{1}{2}\bar{\rho}(\Pi_{ii} + m\tilde{T}^{-1}\Pi_{ii}^\theta)\right] . \tag{33i}$$

The expressions for $\overline{p'u_i''}$ and $\overline{p'd}$ are given by equations (35c) and (45c). Equations (33i) and (31a) can thus be combined to yield the new equation for $\epsilon_d$.

### 14.4. *Compressible Terms:* $\overline{u_i''}, \overline{p'u_i''}, \overline{\rho' u_i''}, \overline{T''}$

To compute these compressibility terms, we begin with the general thermodynamic relations for a polytrope:

$$dQ = cdT \; , \qquad T^{-1}\,dT = (n-1)\rho^{-1}\,d\rho \; , \qquad dS = c_v(n-\gamma)\rho^{-1}\,d\rho \; , \tag{34a}$$

$$n = 1 + m^{-1} = (c_p - c)(c_v - c)^{-1} \; . \tag{34b}$$

The polytropic index $m$ is treated as a free parameter. We recall that

$$\text{adiabatic: } c = 0,\ n = \gamma,\ m = (\gamma - 1)^{-1} \qquad \text{isothermal: } c = \infty,\ n = 1,\ m = \infty \; . \tag{34c}$$

From equation (34a) we derive

$$\frac{p'}{\bar{p}} = n\,\frac{\rho'}{\bar{\rho}} = nm\,\frac{\rho T''}{\bar{\rho}\tilde{T}} \; , \tag{35a}$$

which gives $\bar{p}' = \overline{\rho'} = \overline{\rho T''} = 0$ as required by equation (4c). We also have

$$\bar{p} = \gamma^{-1}c_s^2\,\bar{\rho} \; , \qquad p' = n\gamma^{-1}c_s^2\,\rho' \; , \qquad p' = nm\gamma^{-1}\alpha c_s^2\,\rho T'' \; , \tag{35b}$$

where $\alpha = \tilde{T}^{-1}$. Using the first of equation (5b), we derive the following relations:

$$\overline{u_i''} = -\,\bar{\rho}^{-1}\overline{\rho' u_i''} = -\,m(c_p\tilde{T})^{-1}H_i = -\,m(\gamma - 1)c_s^{-2}H_i \; , \qquad \overline{p'u_i''} = nm(1 - \gamma^{-1})\bar{\rho}H_i \; . \tag{35c}$$

Equations (35c) are consistent with equation (4c) multiplied by $u_i''$ and then averaged. The tensor $B_{ij}$, defined by equation (12c) and which acts as a source of Reynolds stresses, equation (14a), can thus be constructed in terms of known variables. We have

$$B_{ij} = m(\gamma - 1)c_s^{-2}(\delta_{ik}H_j + \delta_{jk}H_i)\bar{p}_{,k} = m(1 - \gamma^{-1})\bar{\rho}H_p^{-1}(\delta_{ik}H_j + \delta_{jk}H_i)\Lambda_k \; , \tag{36a}$$

which makes equation (14a) easier to compare with the incompressible case. Since in a region of unstable stratification $H_i > 0$, $B_{ij}$ must be positive, and since $\Lambda_k < 0$, equation (23e), it follows that

$$m < 0 \; , \qquad n < 1 \tag{36b}$$

which implies that the fluctuations cannot be isothermal and/or adiabatic. We may further note that all previous incompressible treatments correspond to the case $n = 0, m = -1$.

Next, we consider the terms $\overline{T''}$ that appears in the convective flux equation (24a). Since by definition

$$\overline{T''} = -\,\bar{\rho}^{-1}\overline{\rho' T''} \; , \tag{37a}$$

use of the second relation in equation (35a) gives

$$\overline{T''} = -\,m(\bar{\rho}\tilde{T})^{-1}\langle \rho T''^2\rangle = -\,2m\tilde{T}^{-1}\Psi = -\,2mc_p(\gamma - 1)c_s^{-2}\Psi \; , \tag{37b}$$

where $\Psi$ is given by solving equation (26f). Since $m < 0$, it follows that $\bar{T}'' > 0$. Equation (37b) is the extension of the second term on the right-hand side of equation (57) of Canuto (1992) to the compressible case.

### 14.5. *Dilatation*

From equation (35c), we obtain the dilatation

$$\bar{d} = \frac{\partial}{\partial x_i}\,\bar{u}_i'' = -\,m(\gamma - 1)\,\frac{\partial}{\partial x_i}\,(c_s^{-2}H_i) \; , \tag{37c}$$

which relates the divergence of the mass average turbulent velocity $u_i''$ to the divergence of the ratio of the convective flux to the square of the sound speed.

## 15. PRESSURE CORRELATIONS TERMS

### 15.1. *Temperature-Pressure Correlation*

The third-order moment

$$\Pi_i^\theta \equiv \overline{T''\,\frac{\partial p'}{\partial x_i}} \; , \tag{38a}$$

which enters the equation for the convective flux, equation (24a), will be written as

$$\Pi_i^\theta = \bar{\rho}^{-1}\left\langle \rho T''\,\frac{\partial p'}{\partial x_i}\right\rangle \tag{38b}$$

Substituting $\rho T''$ from equation (35a), we obtain

$$\bar{\rho}\Pi_i^\theta = (2nm)^{-1}\gamma\tilde{T}c_s^{-2}\,\frac{\partial}{\partial x_i}\,\overline{p'^2} \; , \tag{38c}$$

where $\overline{p'^2}$ is given by equation (33i).

### 15.2. *Pressure-Velocity Correlation Tensor* $\Pi_{ij}$

The tensor (eq. [12d])

$$\Pi_{ij} = \overline{u_i'' p'_{,j}} + \overline{u_j'' p'_{,i}} \equiv \Lambda_{ij} + \Lambda_{ji} \tag{39}$$

is one of the most difficult statistics to compute. In compressible turbulence, it has been general practice to adopt the same expression as in the incompressible case, that is (Speziale & Sarkar 1991; Canuto 1992, 1993)

$$\bar{\rho}^{-1}\Pi_{ij} = 2c_4\tau^{-1}b_{ij} + (1-\beta_5)\bar{\rho}^{-1}B_{ij} - \tfrac{4}{5}K(S_{ij} - \tfrac{1}{3}\delta_{ij}S_{kk}) - \alpha_1[b_{ik}S_{jk} + b_{jk}S_{ik} - \tfrac{2}{3}\delta_{ij}S_{pq}b_{pq}] - \alpha_2(b_{ik}V_{jk} + b_{jk}V_{ik}) , \tag{40a}$$

where $B_{ij}$ is defined by equation (12c). Furthermore,

$$b_{ij} = R_{ij} - \tfrac{2}{3}\delta_{ij}K , \qquad 2S_{ij} = \tilde{u}_{i,j} + \tilde{u}_{j,i} , \qquad 2V_{ij} = \tilde{u}_{i,j} - \tilde{u}_{j,i} . \tag{40b}$$

The constants $c_4$, $\alpha_{1,2}$ and $\beta_5$ can be found in the above references.

Since we have no way of assessing the reliability of equation (40a) as a representation of $\Pi_{ij}$ in the case of strong density stratification, we work out a new expression that we derive using the basic equations obtained before. Take the time derivative $D/Dt$ of equation (39) and consider

$$\frac{D}{Dt}\Lambda_{ij} = \left\langle\left(\frac{D}{Dt}u_i''\right)\frac{\partial p'}{\partial x_j}\right\rangle + \left\langle u_i''\frac{\partial}{\partial x_j}\frac{Dp'}{Dt}\right\rangle . \tag{41a}$$

As for the first term, we make use of equation (10b) to eliminate $Du_i''/Dt$ and obtain

$$\left\langle\left(\frac{D}{Dt}u_i''\right)\frac{\partial p'}{\partial x_j}\right\rangle = -\,\tilde{u}_{i,k}\Lambda_{kj} - \left\langle\frac{1}{\rho}\frac{\partial p}{\partial x_i}\frac{\partial p'}{\partial x_j}\right\rangle , \tag{41b}$$

where we have neglected the fourth-order term

$$\left\langle u_k''\frac{\partial}{\partial x_k}u_i''\frac{\partial p'}{\partial x_j}\right\rangle$$

on the grounds that it represents the product of three functions $u_k''$, $u_i''$, and $p'$ that peak at very low wavenumbers, while the integrand weighs more at large $k$'s because of the $k^2$ factor, thus implying possibly a small overlap. Expanding the density as

$$\rho^{-1} = \bar{\rho}^{-1}\left(1 - \frac{\rho'}{\bar{\rho}} + \cdots\right) \tag{41c}$$

and using the pressure scale height $H_p$ and $\Lambda_i$ defined in equation (23e), we obtain

$$\left\langle\frac{1}{\rho}\frac{\partial p}{\partial x_i}\frac{\partial p'}{\partial x_j}\right\rangle = \bar{\rho}^{-1}\left[\left\langle\left(1-\frac{\rho'}{\bar{\rho}}\right)\frac{\partial p'}{\partial x_i}\frac{\partial p'}{\partial x_j}\right\rangle - \frac{1}{2n}H_p^{-1}\Lambda_i\frac{\partial}{\partial x_j}\overline{p'^2}\right] . \tag{41d}$$

In the first term, we neglect $\rho'/\bar{\rho}$ versus unity and take

$$\partial_i\partial_j\overline{p'^2} = 2\overline{\partial_i p'\,\partial_j p'} + 2\overline{p'\,\partial_{ij}^2 p'} \approx 2\overline{\partial_i p'\,\partial_j p'} \tag{41e}$$

since the term we neglect is the product of two functions, one of which ($p'$) peaks at low wavenumbers while the other ($\partial_{ij}^2 p' \sim k^2 p'$) peaks at large wavenumbers, thus implying a small overlap. Thus, equation (41b) becomes

$$\left\langle\left(\frac{D}{Dt}u_i''\right)\frac{\partial p'}{\partial x_j}\right\rangle = -\,\tilde{u}_{i,k}\Lambda_{kj} - \frac{1}{2}\bar{\rho}^{-1}\left[\frac{\partial^2\overline{p'^2}}{\partial x_i\partial x_j} - \frac{1}{n}H_p^{-1}\Lambda_i\frac{\partial}{\partial x_j}\overline{p'^2}\right] . \tag{42}$$

Since $\overline{p'^2}$ is solution of equation (33i), the right-hand side of equation (42) is considered known. Next, we consider the second term in equation (41a),

$$\left\langle u_i''\frac{\partial}{\partial x_j}\frac{Dp'}{Dt}\right\rangle . \tag{43a}$$

First, we employ the second of equation (35b) and subsequently equation (6d) to compute $Dp'/Dt$. We obtain

$$\frac{\gamma}{n}\left\langle u_i''\frac{\partial}{\partial x_j}\frac{Dp'}{Dt}\right\rangle = \phi_{,j}\overline{u_i''\rho'} + \phi A_{ij} - B_i\frac{\partial}{\partial x_j}c_s^2 , \tag{43b}$$

where

$$\phi \equiv c_s^2\,\Phi \tag{43c}$$

$$A_{ij} \equiv \left\langle u_i''\frac{\partial\rho'}{\partial x_j}\right\rangle , \qquad B_i \equiv \left\langle u_i''\frac{\partial}{\partial x_k}\rho u_k''\right\rangle . \tag{43d}$$

In deriving equation (43b) we have made use of an argument similar to the previous one and have neglected a term whose components have a small overlap. $A_{ij}$ is evaluated using the second of equation (35b) with the result

$$A_{ij} = \frac{\gamma}{n} c_s^{-2}\left[\Lambda_{ij} - \overline{u_i'' p'} c_s^{-2} \frac{\partial}{\partial x_j} c_s^2\right]. \tag{44a}$$

As for $B_i$, we first write it as

$$B_i = \frac{\partial}{\partial x_k}(\overline{\rho u_i'' u_k''}) - \left\langle \rho u_k'' \frac{\partial u_i''}{\partial x_k}\right\rangle = \frac{\partial \tau_{ik}}{\partial x_k} - \left\langle \rho u_k'' \frac{\partial u_i''}{\partial x_k}\right\rangle . \tag{44b}$$

At this point, we make the reasonable approximation that the largest contribution to the last term is obtained by taking $\rho \to \bar{\rho}$. The term that remains under the average can then be evaluated by averaging equation (10b), which gives the exact result

$$\left\langle u_k'' \frac{\partial u_i''}{\partial x_k}\right\rangle = -\frac{D}{Dt}\overline{u_i''} - \tilde{u}_{i,k}\overline{u_k''} + (2n\bar{p}\bar{\rho})^{-1}\frac{\partial}{\partial x_i}\overline{p'^2} + \bar{\rho}^{-1}\frac{\partial \tau_{ik}}{\partial x_k} . \tag{44c}$$

Substituting into equation (44b), we obtain

$$B_i = \bar{\rho}\left[\frac{D}{Dt}\overline{u_i''} + \tilde{u}_{i,k}\overline{u_k''} - (2n\bar{p}\bar{\rho})^{-1}\frac{\partial}{\partial x_i}\overline{p'^2}\right]. \tag{44d}$$

Collecting the results, equation (43b) becomes

$$\left\langle u_i'' \frac{\partial}{\partial x_j}\frac{Dp'}{Dt}\right\rangle = \overline{u_i'' p'}\frac{\partial \Phi}{\partial x_j} + \Phi\Lambda_{ij} - \frac{n}{\gamma}\bar{\rho}\left[\frac{D}{Dt}\overline{u_i''} + \tilde{u}_{i,k}\overline{u_k''} - (2n\bar{p}\bar{\rho})^{-1}\frac{\partial}{\partial x_i}\overline{p'^2}\right]\frac{\partial c_s^2}{\partial x_j} . \tag{44e}$$

Using equations (42) and (44e), equation (41a) then becomes

$$\frac{D}{Dt}\Lambda_{ij} = \Phi\Lambda_{ij} - \tilde{u}_{i,k}\Lambda_{kj} + \overline{u_i'' p'}\frac{\partial \Phi}{\partial x_j} - \frac{n}{\gamma}\bar{\rho}\left[\frac{D}{Dt}\overline{u_i''} + \tilde{u}_{i,k}\overline{u_k''}\right]\frac{\partial}{\partial x_j}c_s^2 + \frac{1}{2}c_s^2\bar{\rho}^{-1}\left[\frac{1}{nH_p}\Lambda_i a_j - \frac{\partial a_i}{\partial x_j}\right], \tag{45a}$$

where

$$a_i \equiv c_s^{-2}\frac{\partial}{\partial x_i}\overline{p'^2} . \tag{45b}$$

Once $\Lambda_{ij}$ is known, the pressure-velocity correlation tensor $\Pi_{ij}$ is also known, equation (39), and so is the pressure-dilation term since

$$\overline{p'd} = \frac{\partial}{\partial x_i}(\overline{p'u_i''}) - \frac{1}{2}\Pi_{ii} , \tag{45c}$$

where $\overline{p'u_i''}$ is given by equation (35c).

### 15.3. *The Terms* $\langle u_i''(Dp'/Dt)\rangle$ *and* $\langle T''(Dp'/Dt)\rangle$

These two terms appear in the dynamic equation for the convective fluxes, equation (24a), and in the temperature variance, equation (26f). To compute the first term, we employ equation (35b) to write $p'$ in terms of $\rho'$, equation (6d) for $D\rho'/Dt$ as well as the definition (eq. [33b]). We obtain

$$\overline{u_i''\frac{Dp'}{Dt}} = \frac{n}{\gamma}c_s^2[\Phi\overline{u_i''\rho'} - B_i] , \tag{46a}$$

where $B_i$ is defined in equation (43d) and is given by equation (44d). As for the second quantity, we use the same procedure to obtain

$$\overline{T''\frac{Dp'}{Dt}} = -\frac{n}{\gamma}c_s^2\left[\bar{\rho}\overline{T''}\Phi + \overline{T''\frac{\partial}{\partial x_j}\rho u_j''}\right]. \tag{46b}$$

To treat the last term in equation (46b), we first approximate it as

$$\overline{T''\frac{\partial}{\partial x_j}\rho u_j''} \approx \bar{\rho}^{-1}\overline{\rho T''\frac{\partial}{\partial x_j}\rho u_j''} \tag{46c}$$

and then use the second expression for $p'$ of equation (35b). The resulting expression is the same as that in equation (33c). Thus, finally,

$$\overline{T''\frac{Dp'}{Dt}} = -\frac{n}{\gamma}c_s^2\left\{\bar{\rho}\Phi\overline{T''} + \frac{1}{mn}\left(\frac{\tilde{T}}{\bar{p}}\right)\left[\frac{\partial}{\partial x_i}\bar{\rho}(\overline{p'u_i''} + nm^2\bar{p}\tilde{T}^{-2}\Psi_i) - \frac{1}{2}\bar{\rho}(\Pi_{ii} + m\tilde{T}^{-1}\Pi_{ii}^{\theta})\right]\right\} . \tag{46d}$$

## 16. NONLOCALITY. DIFFUSION, THIRD-ORDER MOMENTS

To account for nonlocality, we have to compute the three diffusion terms entering equations (14a), (14d), (24a), (23c), (26a), and (26b), that is, the third-order moments,

$$\tau_{ijk} \equiv \overline{\rho u_i'' u_j'' u_k''} , \qquad \tau_{ijk} = \bar{\rho} R_{ijk} \tag{47a}$$

$$h_{ij} \equiv \overline{\rho u_i'' u_j'' T''} , \qquad h_{ij} = \bar{\rho} H_{ij} \tag{47b}$$

$$\psi_i \equiv \overline{\rho u_i'' T''^2} , \qquad \psi_i = \bar{\rho} \Psi_i \tag{47c}$$

$$\lambda \equiv \langle \rho'^3 \rangle , \qquad \lambda \equiv \bar{\rho}^3 \sigma . \tag{47d}$$

The last third-order moment will be shown to be implied by the equation for $\psi_i$.

### 16.1. *The Tensor* $\tau_{ijk} \equiv \overline{\rho u_i'' u_j'' u_k''} = \bar{\rho} R_{ijk}$

To derive the dynamic equation for this variable, we multiply equation (7a) by $u_j u_k$ and equation (11a) by $u_k$. Summing the results and rearranging, we obtain

$$\frac{\partial}{\partial t} (\rho u_i u_j u_k) + \frac{\partial}{\partial x_m} (\rho u_i u_j u_k u_m) = F_i u_j u_k + F_j u_k u_i + F_k u_i u_j . \tag{48a}$$

We have averaged equation (48a) and derived an equation for $R_{ijk}$. However, the final form is not very transparent. Thus, we have decided to follow another route. We take the $D/Dt$ of $\tau_{ijk}$. Eliminating the $D\rho/Dt$ term via equation (6a), we obtain

$$\frac{D}{Dt} \tau_{ijk} = - \tilde{u}_{m,m} \tau_{ijk} - \left\langle u_i'' u_j'' u_k'' \frac{\partial}{\partial x_m} (\rho u_m'') \right\rangle + \rho \left\langle \left[ u_i'' u_j'' \frac{D u_k''}{Dt} + \text{perm.} \right] \right\rangle , \tag{49a}$$

where perm. indicates that one must add two more terms with running indices *jki* and *kij*. We also recall that $\langle A \rangle \equiv \bar{A}$. In each of the three terms in the square brackets, we substitute equation (10b). We obtain

$$\left\langle \left( u_i'' u_j'' \frac{D u_k''}{Dt} + \text{perm.} \right) \right\rangle = - (\tau_{ijm} \tilde{u}_{k,m} + \text{perm.}) + \frac{1}{\bar{\rho}} \left( \tau_{ij} \frac{\partial}{\partial x_m} \tau_{km} + \text{perm.} \right) - A_{ijk} - B_{ijk} . \tag{49b}$$

Here, perm. means the addition of two terms with the indices *ijk* permuted (the dummy index *m* is unchanged). The last two terms in equation (49b) are given by

$$A_{ijk} \equiv \left\langle \rho u_i'' u_i'' u_m'' \frac{\partial u_k''}{\partial x_m} \right\rangle + \text{perm.} \tag{49c}$$

$$B_{ijk} \equiv \left( \overline{u_i'' u_j'' F_k} - \frac{1}{\bar{\rho}} \overline{\rho u_i'' u_j''} \bar{F}_k \right) + \text{perm.} \tag{49d}$$

Let us compute the last term. We have

$$\overline{u_i'' u_j'' F_k} - \frac{1}{\bar{\rho}} \overline{\rho u_i'' u_j''} \bar{F}_k = \overline{u_i'' u_j'' F_k'} - \frac{1}{\bar{\rho}} \overline{\rho' u_i'' u_j''} \bar{F}_k . \tag{49e}$$

Next, we substitute the definition of $F_i$, equations (7a) and (7b) and neglect the viscous term. We derive

$$\overline{u_i'' u_j'' F_k} - \frac{1}{\bar{\rho}} \overline{\rho u_i'' u_j''} \bar{F}_k = - \pi_{ijk} + g \bar{\rho} \Lambda_k \frac{1}{\bar{\rho}} \overline{\rho' u_i'' u_j''} , \tag{49f}$$

where

$$\pi_{ijk} \equiv \left\langle u_i'' u_j'' \frac{\partial p'}{\partial x_k} \right\rangle . \tag{49g}$$

Finally, we employ the second relation in equation (35a) to rewrite the last term in equation (49f) as

$$g \bar{\rho} \Lambda_k \frac{1}{\bar{\rho}} \overline{\rho' u_i'' u_j''} = mg \tilde{T}^{-1} h_{ij} \Lambda_k , \tag{49h}$$

where we have used the definition (47b). Substituting equation (49h) into equation (49f) and then into equation (49d), equation (49a) becomes

$$\frac{D}{Dt} \tau_{ijk} + \tilde{u}_{m,m} \tau_{ijk} + \frac{\partial}{\partial x_m} (\overline{\rho u_i'' u_j'' u_k'' u_m''}) = - (\tau_{ijm} \tilde{u}_{k,m} + \text{perm.}) + \frac{1}{\bar{\rho}} \left( \tau_{ij} \frac{\partial}{\partial x_m} \tau_{km} + \text{perm.} \right) + mg \tilde{T}^{-1} [h_{ij} \Lambda_k + \text{perm.}] - \Pi_{ijk} , \tag{50a}$$

where

$$\Pi_{ijk} = \pi_{ijk} + \text{perm.} \tag{50b}$$

Using equations (47a), (47b), and the tensor

$$\overline{\rho u_i'' u_i'' u_k'' u_m''} \equiv \bar{\rho} R_{ijkm} \tag{50c}$$

as well as equation (6c), (50a) becomes

$$\bar{\rho} \frac{D}{Dt} R_{ijk} + \frac{\partial}{\partial x_m} \bar{\rho} R_{ijkm} = -\bar{\rho}(R_{ijm} \tilde{u}_{k,m} + \text{perm.}) + \left[ R_{ij} \frac{\partial}{\partial x_m} (\bar{\rho} R_{km}) + \text{perm.} \right] + mg\tilde{T}^{-1}\bar{\rho}(H_{ij}\Lambda_k + \text{perm.}) - \Pi_{ijk} . \tag{50d}$$

Next, it has been known for many years that one cannot assume that the fourth-order moments are Gaussian distributed:

$$R_{ijkm} = R_{ij} R_{km} + R_{ik} R_{jm} + R_{im} R_{jk} + C_{ijkm} \tag{50e}$$

with zero cumulant

$$C_{ijkm} = 0 , \tag{50f}$$

since that leads to negative energies. The pragmatic solution adopted in all treatments is to make sure that there is no accumulation of the third-order moments by assuming that the effect of nonzero cumulant $C_{ijkm}$ is represented by a damping timescale $\tau$. Thus, equation (50d) becomes

$$\begin{aligned} \left(\frac{D}{Dt} + \tau^{-1}\right) R_{ijk} = &-(R_{ijm} \tilde{u}_{k,m} + \text{perm.}) - \left( R_{im} \frac{\partial}{\partial x_m} R_{jk} + R_{jm} \frac{\partial}{\partial x_m} R_{ki} + R_{km} \frac{\partial}{\partial x_m} R_{ij} \right) \\ &+ mg\tilde{T}^{-1}(H_{ij}\Lambda_k + \text{perm.}) - (\bar{\rho})^{-1}\Pi_{ijk} . \end{aligned} \tag{50g}$$

Finally, we have to work out an expression for the pressure term $\Pi_{ijk}$. In all incompressible treatments, this term is not actually computed; rather, it is assumed phenomenologically that it entails a relaxation process with a timescale $\tau$, that is,

$$\Pi_{ijk} \sim -\tau^{-1} R_{ijk} . \tag{50h}$$

There is little justification for such an assumption in the case of compressible turbulence not ultimately because it is not at all obvious which timescale to use since we have

$$\tau,\ \tau_a \text{ and } \tau_s \equiv (U_{i,j} U_{j,i})^{-1/2} , \tag{50i}$$

where the first two $\tau$'s have already been defined.

We shall proceed in the following way. First, we use equation (35a) to write

$$\pi_{ijk} \equiv \left\langle u_i'' u_j'' \frac{\partial p'}{\partial x_k} \right\rangle = nmc_p(1-\gamma^{-1}) \left\langle u_i'' u_j'' \frac{\partial}{\partial x_k} \rho T'' \right\rangle = nmc_p(1-\gamma^{-1}) \left[ \left\langle u_i'' u_j'' T'' \frac{\partial \rho}{\partial x_k} \right\rangle + \left\langle \rho u_i'' u_j'' \frac{\partial T''}{\partial x_k} \right\rangle \right] . \tag{51a}$$

In the first term we shall take $\rho \to \bar{\rho}$, and so

$$\pi_{ijk} = nmc_p(1-\gamma^{-1}) \left[ \bar{\rho}^{-1} \frac{\partial \bar{\rho}}{\partial x_k} h_{ij} + \left\langle \rho u_i'' u_j'' \frac{\partial T''}{\partial x_k} \right\rangle \right] . \tag{51b}$$

Since

$$\frac{\partial}{\partial x_k} h_{ij} = \left\langle T'' \frac{\partial}{\partial x_k} \rho u_i'' u_j'' \right\rangle + \left\langle \rho u_i'' u_j'' \frac{\partial T''}{\partial x_k} \right\rangle , \tag{51c}$$

we further have

$$\pi_{ijk} = nmc_p(1-\gamma^{-1}) \left[ \bar{\rho}^{-1} \frac{\partial \bar{\rho}}{\partial x_k} h_{ij} + \frac{\partial}{\partial x_k} h_{ij} - \left\langle T'' \frac{\partial}{\partial x_k} \rho u_i'' u_j'' \right\rangle \right] . \tag{51d}$$

Finally, by approximating the last term, we have

$$\pi_{ijk} = nmc_p(1-\gamma^{-1}) \left[ \bar{\rho}^{-1} \frac{\partial \bar{\rho}}{\partial x_k} h_{ij} + \frac{\partial}{\partial x_k} h_{ij} - \overline{T''} \frac{\partial}{\partial x_k} \tau_{ij} \right] . \tag{51e}$$

An alternative expression derivable from equation (51b) is

$$\pi_{ijk} = nmc_p(1-\gamma^{-1}) \left[ \bar{\rho}^{-1} \frac{\partial \bar{\rho}}{\partial x_k} h_{ij} + \tau_{ij} \frac{\partial \overline{T''}}{\partial x_k} \right] . \tag{51f}$$

Expressions (51e) and (51f) can be rewritten as

$$\bar{\rho}^{-1}\pi_{ijk} = nmc_p(1-\gamma^{-1}) \left[ (2H_{ij} - R_{ij}\overline{T''})\bar{\rho}^{-1} \frac{\partial \bar{\rho}}{\partial x_k} + \frac{\partial H_{ij}}{\partial x_k} - \overline{T''} \frac{\partial R_{ij}}{\partial x_k} \right] \tag{51g}$$

$$\bar{\rho}^{-1}\pi_{ijk} = nmc_p(1-\gamma^{-1})\left[\bar{\rho}^{-1}\frac{\partial\bar{p}}{\partial x_k}H_{ij} + R_{ij}\frac{\partial\bar{T}''}{\partial x_k}\right]. \tag{51h}$$

Equation (50f) can be compared term by term with the analogous equation for the fully incompressible case equation (55a) of (Canuto 1992).

16.2. *The Tensor* $h_{ij} \equiv \overline{\rho u_i'' u_j'' T''} = \bar{\rho}H_{ij}$

To derive the dynamic equation for this tensor, we can follow two procedures. We can multiply equation (11a) by $T$ and equation (21d) by $u_i u_j$. Summing the two and rearranging, we obtain

$$\frac{\partial}{\partial t}(\rho T u_i u_j) + \frac{\partial}{\partial x_k}(\rho T u_i u_j u_k) = T(F_i u_j + u_j F_i) + u_i u_j(A, B) . \tag{52a}$$

Next, we average equation (52a). We have carried out this process only to end up with an equation that is quite cumbersome and, more to the point, rather difficult to interpret physically. We thus decided to use a different approach. We begin by writing

$$\frac{D}{Dt}h_{ij} = \left\langle \rho T''\frac{D}{Dt}u_i'' u_j'' \right\rangle + \left\langle u_i'' u_j''\frac{D}{Dt}\rho T'' \right\rangle \equiv A_{ij} + B_{ij} . \tag{53a}$$

From equation (10b) it follows that

$$\frac{D}{Dt}u_i'' u_j'' = -[\tilde{u}_{i,k}u_j'' u_k'' + \tilde{u}_{j,k}u_i'' u_k''] - [u_{i,k}'' u_j'' u_k'' + u_{j,k}'' u_i'' u_k''] + (\gamma_i u_j'' + \gamma_j u_i'') , \tag{53b}$$

where

$$\gamma_i \equiv \rho^{-1}F_i - (\bar{\rho})^{-1}\bar{F}_i + (\bar{\rho})^{-1}\frac{\partial}{\partial x_k}\tau_{ik} . \tag{53c}$$

Thus, we obtain

$$A_{ij} \equiv -(\tilde{u}_{i,k}h_{jk} + \tilde{u}_{j,k}h_{ik}) - \left\langle \rho T'' u_k''\frac{\partial}{\partial x_k}u_i'' u_j'' \right\rangle + \langle \rho T''(\gamma_i u_j'' + \gamma_j u_i'')\rangle . \tag{53d}$$

Next, we work out the last two terms in equation (53d). Using the definitions of $\gamma_i$ and that of $F_i$, equations (7b) and (7c) but neglecting the viscous terms, we obtain

$$\langle \rho T''(\gamma_i u_j'' + \gamma_j u_i'')\rangle = c_p^{-1}\left(H_i\frac{\partial}{\partial x_k}\tau_{jk} + H_j\frac{\partial}{\partial x_k}\tau_{ik}\right) + \left\langle \rho T'' u_j''\left\{\frac{1}{\langle\rho\rangle}\frac{\partial\bar{p}}{\partial x_i} - \frac{1}{\rho}\frac{\partial p}{\partial x_i}\right\}\right\rangle + \langle i \to j\rangle . \tag{53e}$$

In the curly brackets, the two largest terms give

$$\{\cdots\} = g\Lambda_i\frac{\rho'}{\langle\rho\rangle} - (\bar{\rho})^{-1}\frac{\partial p'}{\partial x_i} . \tag{53f}$$

Thus, writing $\rho'$ as from equation (35a), we obtain

$$\left\langle \rho T'' u_j''\left\{\frac{1}{\langle\rho\rangle}\frac{\partial\bar{p}}{\partial x_i} - \frac{1}{\rho}\frac{\partial p}{\partial x_i}\right\}\right\rangle + \langle i \to j\rangle = mg\tilde{T}^{-1}(\Lambda_i\psi_j + \Lambda_j\psi_i) - \Pi_{ij}^{\theta} , \tag{53g}$$

where the pressure-temperature correlation tensor is defined as

$$\Pi_{ij}^{\theta} \equiv \left\langle u_i'' T''\frac{\partial p'}{\partial x_j}\right\rangle + \left\langle u_j'' T''\frac{\partial p'}{\partial x_i}\right\rangle \equiv \pi_{ij}^{\theta} + \pi_{ji}^{\theta} , \tag{53h}$$

and $\psi_i$ is defined in equation (47c). The tensor $A_{ij}$ cannot be further reduced. Next, consider $B_{ij}$ and in particular

$$\frac{D}{Dt}\rho T'' = m^{-1}\frac{D}{Dt}\rho'\tilde{T} = m^{-1}\tilde{T}\left(\rho'\Phi - \frac{\partial}{\partial x_k}\rho u_k''\right) , \tag{54a}$$

where we have used equations (35a) and then (6d). We recall that $\Phi$ is defined in equation (33b). Thus,

$$B_{ij} = h_{ij}\Phi - m^{-1}\tilde{T}\left\langle u_i'' u_j''\frac{\partial}{\partial x_k}\rho u_k''\right\rangle . \tag{54b}$$

In equation (53a) we add to both sides the divergence

$$\frac{\partial}{\partial x_k}(\overline{\rho T'' u_i'' u_j'' u_k''}) \tag{54c}$$

and consider the combination

$$C_{ij} \equiv \frac{\partial}{\partial x_k}(\overline{\rho T'' u_i'' u_j'' u_k''}) - \left\langle \rho T'' u_k'' \frac{\partial}{\partial x_k} u_i'' u_j'' \right\rangle - m^{-1}\tilde{T}\left\langle u_i'' u_j'' \frac{\partial}{\partial x_k} \rho u_k'' \right\rangle , \tag{55a}$$

where the second term comes from $A_{ij}$, equation (53d), while the last term comes from $B_{ij}$, equation (54b). We have

$$C_{ij} = \left\langle u_i'' u_j'' \frac{\partial}{\partial x_k}(\rho T'' u_k'') \right\rangle - m^{-1}\tilde{T}\left\langle u_i'' u_j'' \frac{\partial}{\partial x_k} \rho u_k'' \right\rangle . \tag{55b}$$

Next, using equation (23g), we write the first term as

$$c_p^{-1} r_{ij} \frac{\partial}{\partial x_k}(\bar{\rho} H_k) . \tag{55c}$$

In the second term in equation (55b), we substitute $\rho = \bar{\rho} + \rho'$. Using equation (5b) to write $\langle \rho' u_k'' \rangle$, we derive

$$\left\langle u_i'' u_j'' \frac{\partial}{\partial x_k} \rho u_k'' \right\rangle = \bar{\rho}_{,k} R_{ijk} - r_{ij} \frac{\partial}{\partial x_k}(\bar{\rho} \overline{u_k''}) , \tag{55d}$$

which finally gives

$$C_{ij} = c_p^{-1} r_{ij} \frac{\partial}{\partial x_k}(\bar{\rho} H_k) - m^{-1}\tilde{T}\left[ \bar{\rho}_{,k} R_{ijk} - r_{ij} \frac{\partial}{\partial x_k}(\bar{\rho} \overline{u_k''}) \right] . \tag{55e}$$

The dynamic equation (53a) for $h_{ij}$ then becomes

$$\frac{D}{Dt} h_{ij} + D_f = h_{ij}\Phi - (\tilde{u}_{i,k} h_{jk} + \tilde{u}_{j,k} h_{ik}) + c_p^{-1} r_{ij} \frac{\partial}{\partial x_k}(\bar{\rho} H_k) + c_p^{-1}\left( H_i \frac{\partial}{\partial x_k} \tau_{jk} + H_j \frac{\partial}{\partial x_k} \tau_{ik} \right)$$

$$+ mg\tilde{T}^{-1}(\Lambda_i \psi_j + \Lambda_j \psi_i) - m^{-1}\tilde{T}\left[ \bar{\rho}_{,k} R_{ijk} - r_{ij} \frac{\partial}{\partial x_k}(\bar{\rho} \overline{u_k''}) \right] - \Pi_{ij}^{\theta} , \tag{56a}$$

where the diffusion term is defined by equation (54c) as

$$D_f \equiv \frac{\partial}{\partial x_k}(\overline{\rho T'' u_i'' u_j'' u_k''}) . \tag{56b}$$

Next, we approximate the fourth-order moment as before, namely

$$\overline{\rho T'' u_i'' u_j'' u_k''} \to c_p^{-1}[H_i \tau_{jk} + H_j \tau_{ik} + H_k \tau_{ij}] + C_{ijk} , \tag{57a}$$

where the cumulant $C$ is nonzero. Equation (56a) then becomes

$$\left( \frac{D}{Dt} + \tau^{-1} \right) h_{ij} = h_{ij}\Phi - (\tilde{u}_{i,k} h_{jk} + \tilde{u}_{j,k} h_{ik}) - c_p^{-1}\bar{\rho}[R_{ik} H_{j,k} + R_{jk} H_{i,k} + H_k R_{ij,k}]$$

$$+ mg\tilde{T}^{-1}(\Lambda_i \psi_j + \Lambda_j \psi_i) - m^{-1}\tilde{T}\left[ \bar{\rho}_{,k} R_{ijk} - r_{ij} \frac{\partial}{\partial x_k}(\bar{\rho} \overline{u_k''}) + H_{ij} \frac{\partial}{\partial x_k}(\bar{\rho} H_k) \right] - \Pi_{ij}^{\theta} \tag{57b}$$

To simplify equation (57b) further, we use equations (47b) and (47c) and obtain

$$\left( \frac{D}{Dt} + \tau^{-1} \right) H_{ij} = H_{ij} c_s^{-2} \frac{D}{Dt} c_s^2 - (\tilde{u}_{i,k} H_{jk} + \tilde{u}_{j,k} H_{ik}) + mg\tilde{T}^{-1}(\Lambda_i \Psi_j + \Lambda_j \Psi_i) - c_p^{-1}[R_{ik} H_{j,k} + R_{jk} H_{i,k} + H_k R_{ij,k}]$$

$$- \bar{\rho}^{-1}\Pi_{ij}^{\theta} - m^{-1}\tilde{T}\bar{\rho}^{-1}\left[ \bar{\rho}_{,k} R_{ijk} - r_{ij} \frac{\partial}{\partial x_k}(\bar{\rho} \overline{u_k''}) + H_{ij} \frac{\partial}{\partial x_k}(\bar{\rho} H_k) \right] . \tag{58a}$$

Equation (58a) can now be compared directly with the incompressible case, equation (55b) of Canuto (1992). As in the previous case, we shall not use a phenomenological expression for $\Pi_{ij}^{\theta}$ but rather derive an expression within the formalism itself. We begin by using equation (35a) to write

$$\pi_{ij}^{\theta} = \left\langle u_i'' T'' \frac{\partial p'}{\partial x_j} \right\rangle = nm(1 - \gamma^{-1}) c_p \left\langle u_i'' T'' \frac{\partial}{\partial x_j} \rho T'' \right\rangle = nm(1 - \gamma^{-1}) c_p \left[ \left\langle u_i'' T''^2 \frac{\partial \rho}{\partial x_j} \right\rangle + \left\langle u_i'' \rho T'' \frac{\partial T''}{\partial x_j} \right\rangle \right] , \tag{58b}$$

which we approximate as

$$\pi_{ij}^{\theta} = nm(1 - \gamma^{-1}) c_p \left[ \bar{\rho}^{-1} \bar{\rho}_{,j} \psi_i + \left\langle u_i'' \rho T'' \frac{\partial T''}{\partial x_j} \right\rangle \right] . \tag{58c}$$

The last term will be approximated as in equation (51f) so that

$$\pi_{ij}^{\theta} = nm(1 - \gamma^{-1}) c_p \bar{\rho}[\bar{\rho}^{-1} \bar{\rho}_{,j} \Psi_i + c_p^{-1} H_i \tilde{T}_{,j}''] . \tag{58d}$$

An alternative procedure to compute the last term in equation (58b) is by way of taking $\rho \to \bar{\rho}$ and considering that

$$\frac{\partial}{\partial x_j}\,\psi_i \approx \bar{\rho}\overline{u_i''\,\frac{\partial T''^2}{\partial x_j}} + \frac{\partial \bar{\rho}}{\partial x_j}\,\overline{u_i''\,T''^2} + \bar{\rho}\,\overline{T''^2\,\frac{\partial}{\partial x_j}\,u_i''} \approx \bar{\rho}\overline{u_i''\,\frac{\partial T''^2}{\partial x_j}} + \frac{\partial \bar{\rho}}{\partial x_j}\,\Psi_i + 2\bar{\rho}\Psi\,\frac{\partial}{\partial x_j}\,\overline{u_i''}\ ,$$

so that once substituted in equation (58b) gives

$$\pi^{\theta}_{ij} = nm(1-\gamma^{-1})c_p\,\bar{\rho}[\bar{\rho}^{-1}\bar{\rho}_{,j}\Psi_i + \tfrac{1}{2}\Psi_{i,j} - \Psi\bar{u}''_{i,j}]\ . \tag{58e}$$

Interchanging the indices $i \to j$ and summing the results, one obtains the desired expression for $\Pi^{\theta}_{ij}$, equation (53h).

### 16.3. *The Vector* $\psi_i \equiv \overline{\rho u_i''\,T''^2} = \bar{\rho}\Psi_i$

In order to derive the dynamic equation for this third-order moment, we begin by using equation (35a) to write $\rho T''$ in terms of $\rho'$ so that

$$\psi_i = \overline{\rho u_i''\,T''^2} = (m^{-1}\tilde{T})^2\,\overline{\frac{\rho'^2}{\rho}\,u_i''} \equiv (m^{-1}\tilde{T})^2\Omega_i\ . \tag{59a}$$

Thus, we have

$$\frac{D\Omega_i}{Dt} = -\left\langle \rho^{-2}\,\frac{D\rho}{Dt}\,u_i''\,\rho'^2\right\rangle + \left\langle \rho^{-1}\,\frac{D}{Dt}\,\rho'^2u_i''\right\rangle .$$

Using equation (6a) in the form

$$\frac{D\rho}{Dt} = -\,\rho\tilde{u}_{k,k} - \frac{\partial}{\partial x_i}\,(\rho u_i'')\ , \tag{59b}$$

we obtain

$$\frac{D\Omega_i}{Dt} - \tilde{u}_{k,k}\,\Omega_i = A_i + B_i\ , \tag{59c}$$

where

$$A_i \equiv \left\langle \left(\frac{\rho'}{\rho}\right)^2 u_i''\,\frac{\partial}{\partial x_j}\,(\rho u_j'')\right\rangle \tag{59d}$$

$$B_i \equiv \left\langle \frac{1}{\rho}\,\frac{D}{Dt}\,(\rho'^2u_i'')\right\rangle . \tag{59e}$$

If we further introduce the variable

$$\Omega_i = \bar{\rho}\bar{\Omega}_i\ , \tag{59f}$$

equation (59c) becomes

$$\bar{\rho}\,\frac{D\bar{\Omega}_i}{Dt} = 2\tilde{u}_{k,k}\,\Omega_i + A_i + B_i\ . \tag{59g}$$

Using equations (6d) and (10b), we derive

$$B_i = -\,2\tilde{u}_{k,k}\,\Omega_i - \tilde{u}_{i,j}\,\Omega_j - 2\langle \rho'\rho^{-1}u_i''(\rho u_j'')_{,j}\rangle - \langle \rho'^2\rho^{-1}u_j''\,u_{i,j}''\rangle + \langle \rho^{-1}\rho'^2\gamma_i\rangle\ , \tag{60a}$$

where $\gamma_i$ is given by equation (53c). Equation (59g) then becomes

$$\bar{\rho}\,\frac{D}{Dt}\,\bar{\Omega}_i + D_i = -\,\bar{\rho}\bar{\Omega}_j\,\tilde{u}_{i,j} + \langle \rho^{-1}\rho'^2\gamma_i\rangle + L_i\ , \tag{60b}$$

where the diffusion term is

$$D_i \equiv \frac{\partial}{\partial x_j}\,\langle u_j''\,\rho^{-1}\rho'^2u_i''\rangle\ , \tag{60c}$$

and where $L_i$ is given by

$$L_i = -\,2\left\langle \frac{\rho'}{\rho}\,u_i''(\rho u_j'')_{,j}\right\rangle - \left\langle \frac{\rho'^2}{\rho}\,u_j''\,u_{i,j}''\right\rangle + \left\langle \left(\frac{\rho'}{\rho}\right)^2 u_i''(\rho u_j'')_{,j}\right\rangle + D_i\ . \tag{60d}$$

After some algebra, we obtain without approximations,

$$-\frac{1}{2}\,L_i = \left\langle \frac{\rho'}{\rho}\,u_i''\,\frac{\partial}{\partial x_j}\,(\bar{\rho}u_j'')\right\rangle , \tag{60e}$$

which we further rewrite as

$$-\tfrac{1}{2}L_i = \bar{\rho}[m\tilde{T}^{-1}(\bar{\rho}_{,j}/\bar{\rho})H_{ij} - \overline{u_i''d}] \,, \tag{60f}$$

where we have approximated $\langle \rho'/\rho u_i'' u_{,j}''\rangle$ with $-\overline{u_i''}\bar{d}$. Furthermore, using the definition of $\gamma_i$ and that of $F_i$, equation (7b), we obtain

$$\langle \rho^{-1}\rho'^2\gamma_i\rangle \equiv \bar{\rho}\Gamma_i \qquad \Gamma_i \equiv \bar{\rho}^{-3} < \rho'^2 > \tau_{ij,j} + g\Lambda_i\sigma - \Pi_i^{\theta\theta} \,. \tag{60g}$$

The pressure-density covariance correlation

$$\Pi_i^{\theta\theta} = \left(\frac{1}{\bar{\rho}}\right)^3 \left\langle \rho'^2 \frac{\partial p'}{\partial x_i}\right\rangle \tag{60h}$$

can be computed to be

$$\Pi_i^{\theta\theta} = n\gamma^{-1}\left(\sigma \frac{\partial}{\partial x_i} c_s^2 + \frac{1}{3} c_s^2 \frac{\partial \sigma}{\partial x_i} + c_s^2 \sigma \bar{\rho}^{-1} \frac{\partial \bar{\rho}}{\partial x_i}\right) . \tag{60i}$$

The variable $\sigma$ was defined in equation (47d) and its equation is given below. By adopting the same approximation (50e), we write the diffusion term as

$$\langle u_j'' \rho^{-1}\rho'^2 u_i''\rangle = 2\bar{\rho}\bar{u}_i''\bar{u}_j'' + \tau_{ij}\left\langle\left(\frac{\rho'}{\rho}\right)^2\right\rangle , \tag{61a}$$

where we recall that $\bar{\rho}\bar{u}_i'' = -\overline{\rho' u_i''}$. Equation (60b) then becomes

$$\frac{D\bar{\Omega}_i}{Dt} = -\bar{\Omega}_j\tilde{u}_{i,j} + g\Lambda_i\sigma - \Pi_i^{\theta\theta} - 2m\tilde{T}^{-1}\left(\frac{\bar{\rho}_{,j}}{\bar{\rho}}\right)H_{ij} - R_{ij}\frac{\partial}{\partial x_j}\left\langle\left(\frac{\rho'}{\bar{\rho}}\right)^2\right\rangle - 2\bar{\rho}^{-1}\bar{u}_j''\frac{\partial}{\partial x_j}(\bar{\rho}\overline{u_i''}) \,. \tag{62a}$$

The corresponding incompressible result is equation (55c) of Canuto (1992). In conclusion, since

$$\Psi_i = \left(\frac{\tilde{T}}{m}\right)^2 \bar{\Omega}_i \tag{62b}$$

once $\bar{\Omega}_i$ is known from equation (62a), the vector $\Psi_i$ is also known.

### 16.4. *The Scalar* $\lambda \equiv \langle \rho'^3\rangle = \bar{\rho}^3\sigma$

To derive the dynamic equation for the variable $\sigma$ that enters in equation (62a), we employ equation (6d) to obtain after some algebra

$$\frac{1}{3}\frac{D}{Dt}\overline{\rho'^3} = -\overline{\rho'^3}\tilde{u}_{k,k} - \left\langle \rho'^2 \frac{\partial}{\partial x_j}(\rho u_j'')\right\rangle . \tag{63a}$$

In terms of $\sigma$ we have ($d \equiv u_{i,i}''$)

$$\bar{\rho}^3\frac{D\sigma}{Dt} + D_f = -3\frac{\partial\bar{\rho}}{\partial x_j}\langle \rho'^2 u_j''\rangle - 3\bar{\rho}\langle \rho'^2 d\rangle - 2\langle \rho'^3 d\rangle \,, \tag{63b}$$

where

$$D_f \equiv \frac{\partial}{\partial x_j}\langle u_j''\rho'^3\rangle \,. \tag{63c}$$

If we take

$$\langle \rho'^2 u_j''\rangle \approx \bar{\rho}^2\bar{\Omega}_j \tag{63d}$$

and approximate the fourth-order in equation (63c) in the usual way,

$$\langle u_j''\rho'^3\rangle = 3\overline{\rho'^2}\,\overline{\rho' u_j''} = -3\bar{\rho}\overline{\rho'^2}\,\overline{u_j''} \,, \tag{63e}$$

Equation (63b) becomes

$$\frac{D\sigma}{Dt} = -3\left(\bar{\rho}^{-1}\frac{\partial\bar{\rho}}{\partial x_i}\right)(\bar{\Omega}_i - 3\Gamma\overline{u_i''}) + 3\overline{u_j''}\frac{\partial\Gamma}{\partial x_j} + 3(\Gamma\langle d\rangle - \bar{\rho}^{-2}\langle\rho'^2 d\rangle) - 2\bar{\rho}^{-3}\langle\rho'^3 d\rangle \,, \tag{63f}$$

where $\Gamma \equiv \langle \rho'^2\rangle\bar{\rho}^{-2}$ is obtained as the solution of equation (33i).

## 17. CONCLUSIONS

The model presented here is an attempt to describe compressible, time-dependent, nonlocal turbulent convection in the presence of large-scale flows and arbitrary stratification and radiative forcing. Before this work, compressible turbulence was treated with numerical techniques such as DNS, direct numerical simulations (Passot & Pouquet 1987; Lee et al. 1992; Cambon et al. 1992; Lele 1994; Cabot, Thompson, & Pollak 1995; Huang et al. 1995; Muthsam et al. 1995), and LES, large eddy simulations (Hurburt, Toomre, & Massaguer 1986; Stein & Nordlund 1989; Chan & Sofia 1989; Hossain & Mullan 1991; Cattaneo et al. 1991; Erlebacher, Hussaini, & Speziale 1992; Singh, Roxburgh, & Chan 1994) and with closure models. As for the latter, an attempt was made to apply the DIA two-point closure model to compressible turbulence (Hartke, Canuto, & Alonso 1988), while the one-point closure model was applied in (Rubesin 1989, 1990; Taulbee & VanOsdol 1991; Speziale & Sarkar 1991; Sarkar 1992; Sarkar et al. 1993, 1989; Xiong 1989, 1997; Yoshizawa 1995; Rudiger et al. 1996). The two-point closure approach gives rise to a rather complex set of equations, while the Reynolds stress models suffer from a great many phenomenological inputs that are difficult to assess. To improve the situation, we have developed a self-consistent formalism that does not make use of phenomenological relations. Only one assumption is made: the fluctuations $p'$, $\rho'$, and $\rho T''$ obey a polytropic relation, and that introduces a free parameter, the polytropic index $m$. The major emphasis of previous work (Xiong 1997) was on the treatment of the radiative field, while several turbulence higher order statistics were treated approximately. Here, the emphasis has been on the treatment of turbulence.

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# APPENDIX

## A1. ENTHALPY (CONVECTIVE) FLUX: $H_i = c_p \bar{\rho}^{-1} \overline{\rho u_i'' T''}$

$$\bar{\rho}\left(\frac{D}{Dt} H_i + c_p \frac{\partial}{\partial x_j} H_{ij}\right) = c_p \bar{\rho} R_{ij} \beta_j - \bar{\rho} H_j \tilde{u}_{i,j} - c_p H_{ij}\left[\bar{\rho}_{,j} + \left(\frac{m}{c_p \tilde{T}}\right)\bar{p}_{,j}\right] - c_p[g\bar{\rho}\Lambda_i \overline{T''} + \Pi_i^\theta]$$

$$+ \pi_{ijk}\delta_{jk} + \overline{u_i''}\frac{D\bar{p}}{Dt} + \overline{u_i'' \frac{Dp'}{Dt}} - \overline{u_i'' F'_{j,j}} , \tag{1a}$$

$$\beta_i \equiv -\frac{\partial \tilde{T}}{\partial x_i} + \frac{g}{c_p}\Lambda_i , \qquad \Lambda_i = H_p(\bar{p})^{-1}\frac{\partial \bar{p}}{\partial x_i} , \qquad H_p = \bar{p}(g\bar{\rho})^{-1} , \tag{1b}$$

$$\overline{T''} = -2mc_p(\gamma - 1)c_s^{-2}\Psi , \tag{1c}$$

$$\bar{\rho}\Pi_i^\theta = (2nm)^{-1}\gamma\tilde{T}c_s^{-2}\frac{\partial}{\partial x_i}\overline{p'^2} , \tag{1d}$$

$$\frac{1}{2}\frac{D\overline{p'^2}}{Dt} = \overline{p'^2}\Phi - \frac{n}{\gamma}c_s^2\left[\frac{\partial}{\partial x_i}\bar{\rho}(\overline{p'u_i''} + nm^2\bar{p}\tilde{T}^{-2}\Psi_i) - \frac{1}{2}\bar{\rho}(\Pi_{ii} + m\tilde{T}^{-1}\Pi_{ii}^\theta)\right] , \tag{1e}$$

$$\Phi \equiv c_s^{-2}\frac{D}{Dt}c_s^2 - \tilde{u}_{k,k} , \tag{1f}$$

$$\overline{u_i''\frac{Dp'}{Dt}} = \frac{n}{\gamma}c_s^2(\Phi\overline{u_i''\rho'} - B_i) , \tag{1g}$$

$$B_i = \bar{\rho}\left[\frac{D}{Dt}\overline{u_i''} + \tilde{u}_{i,k}\overline{u_k''} - (2n\bar{p}\bar{\rho})^{-1}\frac{\partial}{\partial x_i}\overline{p'^2}\right] , \tag{1h}$$

$$\overline{u_i''} = -m(\gamma - 1)c_s^{-2}H_i , \qquad \overline{\rho' u_i''} = m(\gamma - 1)\bar{\rho}c_s^{-2}H_i , \qquad \overline{p'u_i''} = nm(1 - \gamma^{-1})\bar{\rho}H_i , \tag{1i}$$

$$\bar{\rho}^{-1}\pi_{ijk} = nmc_p(1 - \gamma^{-1})\left(\bar{\rho}^{-1}\frac{\partial \bar{\rho}}{\partial x_k}H_{ij} + R_{ij}\frac{\partial \overline{T''}}{\partial x_k}\right) . \tag{1j}$$

$$\Pi_{ij}^\theta \equiv \pi_{ij}^\theta - \pi_{ji}^\theta , \tag{2a}$$

$$\pi_{ij}^\theta = nm(1 - \gamma^{-1})c_p\bar{\rho}(\bar{\rho}^{-1}\bar{\rho}_{,j}\Psi_i + \tfrac{1}{2}\Psi_{i,j} - \Psi\bar{u}''_{i,j}) , \tag{2b}$$

$$\Pi_{ij} \equiv \Lambda_{ij} + \Lambda_{ji} \tag{3a}$$

$$\frac{D}{Dt}\Lambda_{ij} = \Phi\Lambda_{ij} - \tilde{u}_{i,k}\Lambda_{kj} + \overline{u_i'' p'}\frac{\partial \Phi}{\partial x_j} - \frac{n}{\gamma}\bar{\rho}\left(\frac{D}{Dt}\overline{u_i''} + \tilde{u}_{i,k}\overline{u_k''}\right)\frac{\partial}{\partial x_j}c_s^2 + \frac{1}{2}c_s^2\bar{\rho}^{-1}\left(\frac{1}{nH_p}\Lambda_i a_j - \frac{\partial a_i}{\partial x_j}\right) , \tag{3b}$$

$$a_i \equiv c_s^{-2}\frac{\partial}{\partial x_i}\overline{p'^2} . \tag{3c}$$

### A2. TEMPERATURE VARIANCE: $\psi \equiv \frac{1}{2}\overline{\rho T''^2} = \bar{\rho}\Psi$

$$c_p \bar{\rho}\left(\frac{D\Psi}{Dt} + \bar{\rho}^{-1} D_f\right) = \bar{\rho} H_i \beta_i + \frac{1}{2}\Pi^\theta_{ii} + \bar{T}'' \frac{D\bar{p}}{Dt} + \overline{T'' \frac{Dp'}{Dt}} - \overline{T'' F^r_{i,i}}$$

$$D_f \equiv \frac{1}{2}\frac{\partial}{\partial x_i}\bar{\rho}\Psi_i \,, \tag{4a}$$

$$\overline{T'' \frac{Dp'}{Dt}} = -\frac{n}{\gamma} c_s^2 \left\{\bar{\rho}\Phi\bar{T}'' + \frac{1}{mn}\left(\frac{\tilde{T}}{\bar{p}}\right)\left[\frac{\partial}{\partial x_i}\bar{\rho}(\overline{p' u_i''} + nm^2 \bar{p}\tilde{T}^{-2}\Psi_i) - \frac{1}{2}\bar{\rho}(\Pi_{ii} + m\tilde{T}^{-1}\Pi^\theta_{ii})\right]\right\}. \tag{4b}$$

### A3. REYNOLDS STRESSES: $R_{ij} = \bar{\rho}^{-1}\overline{\rho u_i'' u_j''}$

$$\bar{\rho}\left(\frac{D}{Dt}R_{ij} + D_{ij}\right) = \Sigma_{ij} + B_{ij} - \pi_{ij} + \delta_{ij}\frac{2}{3}\overline{p'd} - \frac{2}{3}\bar{\rho}\epsilon\,\delta_{ij} \,, \tag{5a}$$

$$D_{ij} = \bar{\rho}^{-1}\frac{\partial}{\partial x_k}\left(\bar{\rho}R_{ijk} + \frac{2}{3}\delta_{ij}\overline{p' u_k''} - \overline{\sigma_{ik} u_j} - \overline{\sigma_{jk} u_i}\right), \tag{5b}$$

$$-\Sigma_{ij} \equiv \bar{\rho}(R_{ik}\tilde{u}_{j,k} + R_{jk}\tilde{u}_{i,k}) \,, \tag{5c}$$

$$B_{ij} = m(1-\gamma^{-1})\bar{\rho}H_p^{-1}(\delta_{ik}H_j + \delta_{jk}H_i)\Lambda_k \,, \tag{5d}$$

$$\pi_{ij} \equiv \Pi_{ij} - \tfrac{1}{3}\delta_{ij}\Pi_{kk} \,, \tag{5e}$$

$$\overline{p'd} = \frac{\partial}{\partial x_i}(\overline{p' u_i''}) - \frac{1}{2}\Pi_{ii} \,. \tag{5f}$$

### A4. THIRD-ORDER MOMENTS

#### A4.1. $R_{ijk} = \bar{\rho}^{-1}\overline{\rho u_i'' u_j'' u_k''}$

$$\left(\frac{D}{Dt} + \tau^{-1}\right)R_{ijk} = -(R_{ijm}\tilde{u}_{k,m} + \text{perm.}) - \left(R_{im}\frac{\partial}{\partial x_m}R_{jk} + R_{jm}\frac{\partial}{\partial x_m}R_{ki} + R_{km}\frac{\partial}{\partial x_m}R_{ij}\right)$$
$$+ mg\tilde{T}^{-1}(H_{ij}\Lambda_k + \text{perm.}) - (\bar{\rho})^{-1}\Pi_{ijk} \,, \tag{6a}$$

#### A4.2. $H_{ij} = \bar{\rho}^{-1}\overline{\rho u_i'' u_j'' T''}$

$$\left(\frac{D}{Dt} + \tau^{-1}\right)H_{ij} = H_{ij}c_s^{-2}\frac{D}{Dt}c_s^2 - (\tilde{u}_{i,k}H_{jk} + \tilde{u}_{j,k}H_{ik}) + mg\tilde{T}^{-1}(\Lambda_i\Psi_j + \Lambda_j\Psi_i) - c_p^{-1}[R_{ik}H_{j,k} + R_{jk}H_{i,k} + H_k R_{ij,k}]$$
$$- \bar{\rho}^{-1}\Pi^\theta_{ij} - m^{-1}\tilde{T}\bar{\rho}^{-1}\left[\bar{\rho}_{,k}R_{ijk} - r_{ij}\frac{\partial}{\partial x_k}(\bar{\rho}\overline{u_k''}) + H_{ij}\frac{\partial}{\partial x_k}(\bar{\rho}H_k)\right]. \tag{7}$$

#### A4.3. $\Psi_i = \bar{\rho}^{-1}\overline{\rho u_i'' T''^2} = (\tilde{T}/m)^2\bar{\Omega}_i$

$$\frac{D\bar{\Omega}_i}{Dt} = -\bar{\Omega}_j\tilde{u}_{i,j} + g\Lambda_i\sigma - \Pi^{\theta\theta}_i - 2m\tilde{T}^{-1}\left(\frac{\bar{\rho}_{,j}}{\bar{\rho}}\right)H_{ij} - R_{ij}\frac{\partial}{\partial x_j}\left(\frac{\overline{\rho'^2}}{\bar{\rho}^2}\right) - 2\bar{\rho}^{-1}\overline{u_j''}\frac{\partial}{\partial x_j}(\bar{\rho}\overline{u_i''}) \,, \tag{8a}$$

$$\Pi^{\theta\theta}_i = n\gamma^{-1}\left(\sigma\frac{\partial}{\partial x_i}c_s^2 + \frac{1}{3}c_s^2\frac{\partial\sigma}{\partial x_i} + c_s^2\sigma\bar{\rho}^{-1}\frac{\partial\bar{\rho}}{\partial x_i}\right). \tag{8b}$$

#### A4.4. $\overline{\rho'^3} = \bar{\rho}^3\sigma$

$$\frac{D\sigma}{Dt} = -3\left(\bar{\rho}^{-1}\frac{\partial\bar{\rho}}{\partial x_i}\right)(\bar{\Omega}_i - 3\Gamma\overline{u_i''}) + 3\overline{u_j''}\frac{\partial\Gamma}{\partial x_j} + 3(\Gamma\langle d\rangle - \bar{\rho}^{-2}\langle\rho'^2 d\rangle) - 2\bar{\rho}^{-3}\langle\rho'^3 d\rangle \,, \tag{9a}$$

$$\Gamma = \bar{\rho}^{-2}\overline{\rho'^2} = n^{-2}\bar{p}^{-2}\overline{p'^2} \,. \tag{9b}$$

### A5. MEAN TEMPERATURE FIELD: $\tilde{T}$

$$\bar{\rho}c_p\frac{D\tilde{T}}{Dt} = -\frac{\partial}{\partial x_i}(F^c_i + \bar{F}^r_i - \overline{p' u_i''}) + \frac{D\bar{p}}{Dt} + \overline{u_i''}\bar{p}_{,i} - \overline{p'd} + \bar{\rho}\epsilon \,. \tag{10a}$$

### A6. LARGE-SCALE VELOCITY FIELD: $\tilde{u}$

$$\bar{\rho}\,\frac{D}{Dt}\,\tilde{u}_i = -\frac{\partial}{\partial x_j}\,(\bar{p}\,\delta_{ij} + \bar{\rho}R_{ij}) - \bar{\rho}g_i\ . \tag{10b}$$

### A7. MEAN DENSITY: $\bar{\rho}$

$$\frac{D}{Dt}\,\bar{\rho} + \bar{\rho}\,\frac{\partial}{\partial x_j}\,\tilde{u}_j = 0\ . \tag{10c}$$

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